A CLASSIFICATION OF IMMERSED KNOTS

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This paper in a sense completes the study begun in [6] and [3]. We shall make extensive use of the notation and results of [6] and shall show that the group of immersed homotopy $n$-spheres in $m$-space fits naturally into exact sequences similar to those of [6]. Alternatively, one can look on this paper as giving geometric meaning to the groups $\pi_nG$ and $\pi_n(G, SO_q)$. (See also [4]. For additional information about immersions see §4 of [3].) In §1 we give the necessary definitions; the main results are stated in §2. We would like to thank the referee for some useful observations.

1. Preliminaries. Throughout this paper all manifolds will be $C^\infty$, compact, and oriented. All maps are transverse to any boundaries and take boundaries to boundaries. If $f: M^m \rightarrow W^m$ is an immersion, we orient the normal bundle of $f$, $\nu_f$, by the equation

$$\tau_M + \nu_f = f^*\tau_W,$$

where $\tau_M$, $\tau_W$ are the tangent bundles of $M$, $W$, respectively. The boundary of $M$, $\partial M$, is oriented by the equation

$$\xi + \tau_{\partial M} = \tau_M | \partial M,$$

where $\xi$ is a line bundle of vectors orthogonal to $\partial M$ with the vectors pointing out from $M$ oriented positively. $-M$ denotes $M$ with the negative orientation. Given a vector bundle $\eta$ over $M$ we shall always identify $M$ with the zero-section of $\eta$; also, we do not distinguish between the normal disk bundle of a submanifold and a tubular neighborhood.

Next, we say that $(f, \bar{\xi}): M^n \rightarrow W^m$ is a framed immersion if $f: M \rightarrow W$ is an immersion and $\bar{\xi}$ is a framing of $\nu_f$, i.e., $\bar{\xi} = (f_1, \cdots, f_{m-n})$ is an ordered collection of orthogonal vector fields of $\nu_f$ (this is compatible with [6] in case $f$ is an imbedding). Now let $M^n$ and $W^m$ be closed manifolds (boundaries are excluded only for simplicity). Two immersions $f$, $g: M \rightarrow W$ are $h$-cobordant if there is an $h$-cobordism $V^{n+1}$ with $\partial V = M \cup -M$ and an immersion $H: V \rightarrow W \times [0, 1]$ with $H|_M = f \times 0$ and $H|_-M = g \times 1$. Two framed immersions $(f, \bar{\xi})$, $(g, \bar{\eta}): M \rightarrow W$ are $h$-cobordant if there is an $h$-cobordism $V^{n+1}$ with $\partial V = M \cup -M$ and a framed immersion $(H, \bar{\xi}C): V \rightarrow W \times [0, 1]$ with $(H, \bar{\xi}C)|_M = (f, \bar{\xi}) \times 0$ and $(H, \bar{\xi}C)|_-M = (g, \bar{\eta}) \times 1$. We use $[M, f]$, $[M, f, \bar{\xi}]$ to denote the $h$-cobordism class of $f$, $(f, \bar{\xi})$, respectively

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the range $W$ will always be clear from the context. Also, let us recall the notion of regular homotopy between immersions (see [5]). Two framed immersions $(f, \mathfrak{F}), (g, \mathfrak{G}): M \to W$ will be called regularly homotopic if there is a 1-parameter family of framed immersions $(H_t, \mathfrak{C}_t): M \to W$ with $(H_0, \mathfrak{C}_0) = (f, \mathfrak{F})$ and $(H_1, \mathfrak{C}_1) = (g, \mathfrak{G})$.

As usual, $D^n$ will be the closed unit ball in Euclidean $n$-space $\mathbb{R}^n$ with the natural orientation and $S^{n-1} = \partial D^n$. $G_n$ denotes the $H$-space of maps of $S^{n-1} \to S^{n-1}$ of degree $+1$ and $SO_n$ will be the subspace of orthogonal maps. The natural inclusion $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ gives rise to inclusions $D^n \subseteq D^{n+1}$, $G_n \subseteq G_{n+1}$, $SO_n \subseteq SO_{n+1}$. Let $G = \lim_{n \to \infty} G_n$ and $SO = \lim_{n \to \infty} SO_n$.

2. The exact sequences. For the remainder of this paper we assume that $n \geq 5$ and $k = m - n \geq 3$. We define $I_{m,n}$ to be the set of $h$-cobordism classes of immersed homotopy $m$-spheres in $S^m$ and $I^f_{m,n}$ to be the set of $h$-cobordism classes of framed immersed homotopy $n$-spheres in $S^n$. There is an obvious operation of connected sum which makes $I_{m,n}$ and $I^f_{m,n}$ into abelian groups (see [2, §1.3 and §1.4]). Using [7] it is easy to see that $[\Sigma^n, f] = 0 \in I_{m,n}$ (or $[\Sigma^n, f, \mathfrak{F}] = 0 \in I^f_{m,n}$) if and only if $\Sigma^n$ is the boundary of an $(n+1)$-disk $U^{n+1}$ and there is an immersion $H: U^{n+1} \to D^{m+1}$ (or framed immersion $(H, \mathfrak{C}): U^{n+1} \to D^{m+1}$) so that $H|\Sigma = f$ (or $(H, \mathfrak{C})|\Sigma = (f, \mathfrak{F})$).

Define groups $P_n$ as follows:

$$P_n = \begin{cases} Z, & n \equiv 0 \ (\text{mod } 4) \\ Z_2, & n \equiv 2 \ (\text{mod } 4) \\ 0, & n \text{ odd.} \end{cases}$$

In §3 we shall define homomorphisms $\tilde{\omega}_i, \phi_i$, and $\tilde{\varphi}_i$ making the following sequences exact:

$$(1)_k \cdots \to \pi_n SO_k \to I^m_{n,n} \to \pi_{n-1} SO_k \to I^m_{n-1,n-1} \to \cdots$$

$$(2)_k \cdots \to \pi_n G \to I^m_{n,n} \to \pi_{n-1} G \to \pi_{n-1} SO_k \to \cdots$$

$$(3)_k \cdots \to I^m_{n,n} \to \pi_n (G, SO_k) \to P_n \to I^m_{n-1,n-1} \to \pi_{n-1} (G, SO_k) \to \cdots$$

We also get a commutative (up to sign) diagram:

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Next, let $Im^{m,n}$ be the group of regular homotopy classes of (framed) immersions of $S^n$ in $S^m$. The group operation is again the connected sum in both cases. Then it follows from [5] that there are natural isomorphisms $Im^{m,n} \approx \pi_n V_{m,n} \approx \pi_n (SO, SO_k)$ and $Im^{m,n}_f \approx \pi_n SO_m \approx \pi_n SO$. Finally, we also consider the groups $C^k_n$ and $FC^k_n$ of isotopy classes of imbedding, respectively, framed imbeddings, of $S^n$ in $S^m$ and the groups $\theta^{m,n}$ and $\theta^{m,n}_f$ of $h$-cobordism classes of imbedded, respectively, framed imbedded, homotopy $n$-spheres in $S^m$. Collecting the results of [3], [6], and this paper we get a great many interrelated exact sequences which we shall not bother to write out here. In addition, there are natural suspension maps of the sequences $(l)^{k+N}(l)_{k+1}, l = 1, 2, 3$ and $N \geq 0$, where we take the rear extensions of framings as described in [6, §1.2].

We shall display two interesting diagrams of exact sequences which are derived from standard diagram chasing:

\[ \xymatrix{ \pi_n(G, G_k) \ar[rr]^\theta^{n+1} & & C^n_k \ar[rr]^\theta^{m,n} & & \pi_n(G, G_k) \\ Im^{m,n} \ar[u]_{\theta^{m,n}} \ar[rr]^\theta & & Im^{m,n} \ar[u]_{\theta} & & Im^{m,n} \ar[u]_{\theta} \ar[u]_{\theta} } \]

\[ \xymatrix{ \pi_n S^k = \pi_n (SO_{k+1}, SO_k) \ar[rr]^\pi_{n-1}(SO_{k+1}, SO_k) & & \pi_n S^k \ar[rr]^\pi_{n+1}(SO_{k+1}, SO_k) & & \pi_n S^k } \]

where $\theta^n = \theta^{m+N,n}$ and $I^n = I^{m+N,n}$ for $N > n$. Also, observe that $I_f^{m,n}$ is independent of $m$.

In conclusion, we point out that this paper could have been extended to the case of “relative” immersed knots à la [1].

3. The maps $\tilde{\omega}_i, \phi_i, \bar{\partial}_i$. In this section we shall define the maps $\tilde{\omega}_i, \phi_i,$ and $\bar{\partial}_i$. Let $\omega_i, \phi_i,$ and $\bar{\partial}_i$ be defined as in [6] and let $j_1: \theta^{m,n} \to I^{m,n},$ $j_2: \theta^{m,n}_f \to I^{m,n}_f$ be the obvious maps which assign to each $h$-cobordism class of imbeddings, respectively, framed imbeddings, its corresponding $h$-cobordism class as an immersion, respectively, framed immer-
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Definition

Define

Let \([\Sigma^n, f, \bar{\xi}] \in I^m_n\) and \([N^n, g] \in I^{m-n}\). Define

\[
\phi_1([\Sigma^n, f, \bar{\xi}]) = \phi_2, \quad \phi_3 = \text{composition of} \pi_n(G, SO_k) \rightarrow \pi_n(G, SO) \rightarrow P_n,
\]

\[
d_2 = j_2d_2, \quad d_3 = j_1d_3.
\]

Finally we come to the maps \(\omega_2\) and \(\omega_3\) which, together with the exactness of \((2)_k\) at \(\pi_nG\) and \((3)_k\) at \(\pi_n(G, SO_k)\), are really the heart of this paper.

Lemma 3.1. \(\theta^n_{N,n} \cong I^N_n\), for \(N \geq 2n+1\).

Proof. Now all homotopy \(n\)-spheres \(\Sigma^n\) imbed in \(S^n\) and all immersions \(f: \Sigma \rightarrow S^n\) are regularly homotopic by [5]. A regular homotopy also carries along the framing. Therefore \(j_2\) is onto. That \(j_2\) is one-to-one follows from the fact that we may approximate regular homotopies by imbeddings using Whitney's theorem.

Thus we can define

\[
\omega_2 = \text{composition of} \ I^m_n \rightarrow I^N_n \rightarrow I^N_n \rightarrow \pi_nG,
\]

where \(N \geq 2n+1\).

Now let \([\Sigma^n, f] \in I^{m-n}\). It is well known (see [7]) that \(\Sigma^n\) is obtained by glueing two \(n\)-disks \(D_0\) and \(D_1\) together via a diffeomorphism of their boundaries. We may then take framings \(\xi_i\) of \(\nu_f\) \(D_i\), \(i = 0, 1\), and let \(\alpha_i: D_i \times D^k \rightarrow \nu_f\) \(D_i\) be imbeddings satisfying \(\alpha_i(x, 0) = x\), \(x \in D_i\), and \(d\alpha_i(\xi_i) = \xi_i\), where \(\xi_i\) is the pull-back of a positive frame at \(0 \in D^k\). We can arrange it so that \(\alpha_1(x, y) = \alpha_0(x, \mu(x)y)\), \(x \in \partial D_0\), \(y \in D^k\), for some \(\mu: \partial D_0 \rightarrow SO_k\). (Compare [6, §3.3].) We shall use the notation \(\bar{\xi}_i|\partial D_0 = \mu \bar{\xi}_0|\partial D_0\) to describe this situation. Now assume that \(f|D_0\) is an imbedding and that \(f(\Sigma - D_0)\) does not meet \(f(D_0)\). Let \(f'\) be the composition of \(\Sigma \rightarrow \Sigma^m \rightarrow \pi_n\), for some \(N > n\), and let \(\bar{\xi}_i\) be the framing of \(\nu_{f'}|D_i\) which is the rear ex-
tension of $\Sigma$. Next, move $f'$ into an imbedding $g: \Sigma \to S^{m+N}$ via a regular homotopy $h_t$ satisfying $h_t|D_0 = f'$ and $h_t(\Sigma - D_0) \cap h_t(D_0) = \emptyset$. $h_t$ carries along the framing $\mathfrak{f}_t$, so that we get a framing $\mathfrak{g}_t$ of $S^m$ with $\mathfrak{g}_t|\partial D_0 = \bar{\mu}\mathfrak{g}_0|\partial D_0$, where $\bar{\mu}$ is the composition of $\partial D_0 \to SO_k \to SO_k+N$. If we apply the construction of [6, §3.3], to $(\Sigma, g, \mathfrak{g}_0)$, we get an element $[\lambda] \in \pi_n(G, SO_k)$, i.e., if we let $u:S^{m+N} - g(\Sigma) \to S^{k+N-1}$ be a homotopy inverse of $y \to \alpha_0(x_0, y)$, $x_0 \in D_0$ with $u\alpha_0(x, y) = y$ for all $x \in D_0$, then $\lambda: D_1 \to G_{k+N}$ is given by $\lambda(x)(y) = u\alpha_1(x, y)$, $x \in D_1$, $y \in S^{k+N-1}$. Define $\bar{\omega}_2([\Sigma, f]) = [\lambda]$.

This finishes the definitions of all the maps and it is easy to see, using [6], that they are well defined homomorphisms.

4. Exactness. The proof of exactness of $(1)_k$, $(2)_k$, and $(3)_k$ is very similar to the corresponding proofs given in [6]. In general, the only difference is that here we have immersions instead of imbeddings. Anyone who understands [6] can easily make the appropriate translations. We shall, however, outline a proof of exactness in those places that differ from the corresponding ones in [6]. One essential difference is the fact that any abstract framed surgery can be realized ambiently. Another is that framed immersions of $n$-spheres in $S^m$ are regularly homotopic to immersions in $S^{n+1}$.

We first prove exactness at $\pi_n G$ in $(2)_k$. That $d_2 \bar{\omega}_2 = 0$ follows from [6, §5.5], and the definition of $\bar{\omega}_2$ and $\bar{\phi}_2$. Let $[g] \in \pi_n G$, $g: S^n \to G_N$ and suppose $\bar{\phi}_2([g]) = 0$. Define $\bar{g}: S^n \times S^{N-1} \to S^{N-1}$ by $\bar{g}(x, y) = g(x)(y)$, $x \in S^n$, $y \in S^{N-1}$, and let $\Sigma = \bar{g}^{-1}(e)$, $e \in S^{N-1}$. We may assume that $\Sigma$ is a framed $n$-submanifold of $S^n \times S^{N-1} \subseteq S^{n+N}$. In fact, since $\bar{\phi}_2([g]) = 0$, we may further assume that $\Sigma$ is a homotopy sphere (see [6, §4.7]). By Theorem 6.4 of [5], $\Sigma$ is regularly homotopic to a framed immersion $(\Sigma, f)$ in $S^m$. Then $\bar{\omega}_2([\Sigma, f]) = [g]$.

Next, let us consider exactness at $I^m_n$. There is no problem in showing that $\bar{\omega}_2 \bar{\delta}_2 = 0$. Suppose $[\Sigma, f, \mathfrak{f}] \in I^m_n$ and $\bar{\omega}_2([\Sigma, f, \mathfrak{f}]) = 0$. It follows from the definition of $\bar{\omega}_2$ and the exactness of the Kervaire-Milnor sequence that there is a $\pi$-manifold $W$ and a framing of its stable normal bundle so that $\partial W = \Sigma$ and the framing restricted to $\Sigma$ is essentially a suspension of $\mathfrak{f}$. But then we can use [5] to obtain a framed immersion $(g, \mathfrak{g}): W \to S^m$ so that $g|\Sigma = f$ and $\mathfrak{g}|\Sigma = \mathfrak{f}$. Define $\gamma = \gamma(W, \mathfrak{g}) \in P_{n+1}$ as in §4.5 of [6]. Then $\bar{\delta}_2(\gamma) = [\Sigma, f, \mathfrak{f}]$, because using [5] we can allow in the definition of $\bar{\delta}_2$ not only framed imbeddings of $W$ but also framed immersions.

This finishes our discussion of the exactness of $(2)_k$. Alternatively,
one could observe first that $I^m_n \approx I^n_t \approx \theta^n_t$ using [5], so that exactness follows from the exactness of the Kervaire-Milnor sequence.

The commutativity (up to sign) of (4)$_k$ is proved as in [6] and so by [6, §5.3], the exactness of (3)$_k$ will be established once we show that $\phi_3 \omega_3 = 0$. But consider

If $N>n$, then $j_1$ is an isomorphism (proved similarly to Lemma 3.1), and so $\phi_3 \omega_3 = \phi_3 \omega_3 j_1^{-1} S = 0$.

References

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