NOTE ON NONNEGATIVE MATRICES

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Abstract. Let $A$ be a nonnegative square matrix and $B = D_1AD_2$ where $D_1$ and $D_2$ are diagonal matrices with positive diagonal entries. Several proofs are known for the following theorem: If $A$ is fully indecomposable then $D_1$ and $D_2$ can be chosen so that $B$ is doubly stochastic. Moreover, $D_1$ and $D_2$ are unique up to a scalar factor. It is shown that these results can be easily obtained by considering a minimum of a certain rational function of several variables.

Several recent papers [1], [2], [3], [4] were devoted to the following problem: Given a nonnegative square matrix $A$, find the conditions for the existence of two diagonal matrices $D_1$ and $D_2$ such that $D_1AD_2$ is doubly stochastic. We shall show that it is related to a simple minimum problem. This leads to a short proof of Theorem (6.1) of [1] which avoids the use of Menon's operator.

We begin with some definitions. An $n \times n$ ($n \geq 2$) matrix $A$ is reducible if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where $A_1$ is a $k \times k$ matrix, $1 \leq k \leq n - 1$. Otherwise we say that $A$ is irreducible.

An $n \times n$ ($n \geq 2$) matrix $A$ is fully indecomposable if there do not exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where $A_1$ is a $k \times k$ matrix, $1 \leq k \leq n - 1$.

Theorem. Let $A$ be a nonnegative $n \times n$ fully indecomposable matrix. Then there exist diagonal matrices $D_1$ and $D_2$ with positive diagonals such that $D_1AD_2$ is doubly stochastic. Moreover $D_1$ and $D_2$ are uniquely determined up to scalar multiples.

For the proof we need the following

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**Lemma.** Let $A$ be a nonnegative $n \times n$ matrix. Then $A$ is fully indecomposable if and only if there exist permutation matrices $P$ and $Q$ such that $PAQ$ has a positive main diagonal and is irreducible.

A short proof of this lemma appears in [1].

**Proof of the Theorem.** By the lemma we can assume that $A = (a_{ij})$ has positive main diagonal and is irreducible. Let

$$f(x_1, \cdots, x_n) = \prod_{k=1}^{n} \left( \sum_{i=1}^{n} a_{ki} x_i \right) / \prod_{k=1}^{n} x_k$$

the variables being restricted by

$$(1) \quad x_k > 0 \quad (1 \leq k \leq n), \quad \sum_{k=1}^{n} x_k = 1.$$

Let $(b_i)$ be a boundary point of the region (1) and, for instance, $b_1 = \cdots = b_s = 0, b_k > 0 \ (s < k \leq n)$. Since $A$ is irreducible we infer that at least one entry $a_{ij} > 0 \ for \ 1 \leq i \leq s, s < j \leq n$. This implies that $f(x_1, \cdots, x_n) \to +\infty$ when $(x_i) \to (b_k)$. Therefore $f$ attains its minimum in some point $(c_k)$ of the region (1). The partial derivatives of $f$ vanish at $(c_k)$ since $f$ is homogeneous. Hence,

$$\sum_{k=1}^{n} c_j a_{kj} \left( \sum_{i=1}^{n} a_{ki} c_i \right)^{-1} = 1 \quad (j = 1, \cdots, n)$$

which proves the first assertion of the theorem.

For the uniqueness it is sufficient to prove the following assertion: If the matrices $X = (x_{ij}), D_1 = \text{diag}(d'_1, \cdots, d'_n), D_2 = \text{diag}(d''_1, \cdots, d''_n)$ satisfy

(i) $X$ is irreducible doubly stochastic with positive elements on the main diagonal;

(ii) $d'_i > 0, d''_i > 0 \ (1 \leq i \leq n)$;

(iii) $D_1 XD_1$ is doubly stochastic, then $D_1$ and $D_2$ are scalar matrices.

Since

$$\sum_{j=1}^{n} d'_i x_{ij} d''_j = 1 \Rightarrow (\max d'_i)(\min d''_j) \leq 1,$$

$$\sum_{i=1}^{n} d'_i x_{ij} d''_i = 1 \Rightarrow (\max d'_i)(\min d''_i) \geq 1.$$ we conclude that none of these inequalities is strict. This implies that $x_{rs} = 0$ whenever $d'_r = \max d'_i \ and \ d''_s > \min d''_j \ or \ d'_r < \max d'_i \ and \ d''_s = \min d''_j$. This contradicts (i) unless $D_1$ and $D_2$ are scalar matrices.

The proof is completed.
References


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