A REAL ANALOGUE OF THE GELFAND-NEUMARK THEOREM

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Abstract. Let $A$ be a real Banach $*$-algebra enjoying the following three conditions: $\|x^*x\| = \|x^*\| \|x\|$, $Sp x^*x \geq 0$, and $\|x^*\| = \|x\|$ ($x \in A$). It is shown, after Ingelstam, Palmer, and Behncke, as a real analogue of the Gelfand-Neumark theorem, that $A$ is isometrically $*$-isomorphic onto a real $C^*$-algebra acting on a suitable real (or complex) Hilbert space. The converse is obvious.

The aim of this note is, as a real analogue of the Gelfand-Neumark theorem, to prove the following

Theorem. A real Banach $*$-algebra $A$ is isometrically $*$-isomorphic onto a real $C^*$-algebra acting on a real (or complex) Hilbert space if and only if it satisfies the following three conditions:

1. $\|x^*x\| = \|x^*\| \|x\|$, 
2. $Sp x^*x \geq 0$, and 
3. $\|x^*\| = \|x\|$ ($x \in A$).

The above theorem was conjectured explicitly by Rickart [5, p. 181] and proved by Ingelstam [2] (cf. also Palmer [3], [4] and Behncke [1]). Their proofs were based on complexification of a real Banach $*$-algebra. An alternative proof which we shall give in this note will be done by real $*$-representation on real Hilbert space and by complexification of a real Hilbert space.

Let $A$ be a real Banach $*$-algebra satisfying the conditions stated in the theorem, and $H$ the set of hermitian elements in $A$. Let $R$ be the field of real numbers. In view of (2), the involution is hermitian. Put $\mu(h) = \sup(\lambda; \lambda$ a spectrum of $h$) for all $h$ in $H$. In view of (2), $A$ is symmetric. In view of (3), the involution is continuous. So, we can make use of Rickart [5, Lemma 4.7.10] to get the sublinearity of $\mu$ on $H$, that is,

(i) $\mu(\alpha h) = \alpha \mu(h)$ and
(ii) $\mu(h + k) \leq \mu(h) + \mu(k)$.

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where \(0 \leq \alpha \leq R, h, k \in H\). Owing to the extension theorem of Hahn and Banach, for a fixed element \(a\) in \(A\), there exists a real linear functional, say \(g\), on \(H\) such that \(g(h) = \mu(h)(h \in H)\) and such that \(g((aa^*)^2) = \mu((aa^*)^2)\). Decompose \(x = h + k\), where \(h = (1/2)(x + x^*)\in H\) and \(k = (1/2)(x - x^*)\) being skew adjoint. Put \(f(x) = g(h)\) for all \(x\) in \(A\). Since \(\mu(-x^*x) \leq 0\), we have \(f(x^*x) \geq 0\). Thus, \(f\) is a real state on \(A\).

It is easy to construct a \(\ast\)-representation real Hilbert space and a real \(\ast\)-representation \(\psi\) of \(A\). Moreover, if \(aa^* \neq 0\), \(\psi(a) \neq 0\). Hence, \(\{a; aa^* = 0\}\) is the \(\ast\)-radical of \(A\), that is, the intersection of kernels of all real \(\ast\)-representations of \(A\). In view of (1), the \(\ast\)-radical must be \(\{0\}\). Thus, there exist a \(\ast\)-representation real Hilbert space and a faithful real \(\ast\)-representation of \(A\). Hence, \(A\) is isometrically \(\ast\)-isomorphic onto a real \(C^\ast\)-algebra acting on a real Hilbert space.

In the rest of the if-part proof, we must show that a real \(C^\ast\)-algebra \(A\) acting on a real Hilbert space \(\mathcal{H}\) is isometrically \(\ast\)-isomorphic onto a suitable real \(C^\ast\)-algebra \(A'\) acting on a suitable complex Hilbert space \(\mathcal{H}_C\). Construct \(\mathcal{H}_C\) as the set of formal elements \(x + iy\), where \(x, y \in \mathbb{R}\). Introduce into \(\mathcal{H}_C\) an equality relation: \(x_1 + iy_1 = x_2 + iy_2\) iff \(x_1 = x_2\) and \(y_1 = y_2\) \((x_1, x_2, y_1, y_2 \in \mathcal{H})\), an addition: \(x_1 + iy_1 + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)\) \((x_1, x_2, y_1, y_2 \in \mathcal{H})\), a scalar multiplication: \((a + i\beta)(x + iy) = ax - \beta y + i(\beta x + \alpha y)\) \((a, \beta \in \mathbb{R}, x, y \in \mathcal{H})\), and an inner product:

\[
(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2) + (y_1, y_2) + i((y_1, x_2) - (x_1, y_2))
\]

Then, \(\mathcal{H}_C\) becomes a complex Hilbert space. For each \(a\) in \(A\), we define a mapping \(a' : x + iy \mapsto ax + iay\) \((x, y \in \mathcal{H})\). It is easy to see that \(a'\) is a bounded linear operator acting on \(\mathcal{H}_C\) with \(\|a'\| = \|a\|\). Put \(A' = \{a'; a \in A\}\). The mapping: \(a \mapsto a'\) gives an isometric \(\ast\)-isomorphism of \(A\) onto \(A'\). This completes the if-part proof of the theorem. And the only-if-part proof of the theorem goes as usual fashion.

References

3. T. Palmer, A real \(B^\ast\)-algebra is \(C^\ast\) iff it is hermitian, Notices Amer. Math. Soc. 16 (1969), 222–223. Abstract #663-468.

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