# SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# THE TOEPLITZ-HAUSDORFF THEOREM FOR LINEAR OPERATORS

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THEOREM (TOEPLITZ-HAUSDORFF). *The numerical range* $W(A) = \{(Ax, x): \|x\| = 1, x \in D(A)\}$ *of an arbitrary (perhaps unbounded and not densely defined) linear operator $A$ in a (pre)-Hilbert space (real or complex) is convex.*

PROOF. Since $W(\mu A + \gamma) = \mu W(A) + \gamma$, for scalars $\mu$, $\gamma$, it suffices to consider the situation $(Ax_1, x_1) = 0$, $(Ax_2, x_2) = 1$, $\|x_i\| = 1$, $x_i \in D(A)$, $i = 1, 2$. Let $x = \alpha x_1 + \beta x_2$, $\alpha$ and $\beta$ real, and require

$$\|x\|^2 \equiv \alpha^2 + \beta^2 + 2\alpha\beta \operatorname{Re}(x_1, x_2) = 1, \tag{1}$$

and desire (for each $0 < \lambda < 1$)

$$(Ax, x) \equiv \beta^2 + \alpha\beta\{(Ax_1, x_2) + (Ax_2, x_1)\} = \lambda. \tag{2}$$

Let $B = (Ax_1, x_2) + (Ax_2, x_1)$; if $B$ is real, then the system (1), (2) describes an ellipse (intercepts $\pm 1$, $\pm 1$) and a hyperbola (intercepts $\pm\lambda^{1/2}$) and clearly possesses (four, since $|\operatorname{Re}(x_1, x_2)| < 1$ by Schwarz's Inequality) solutions. But $B$ can always be guaranteed real by using an appropriate (scalar multiple of) $x_1$; i.e., explicitly, use $x_1' = \mu x_1$, where $\mu = a + ib$ satisfies (1′) $|\mu|^2 \equiv a^2 + b^2 = 1$ and (2′) $\operatorname{Im} B(x_1') \equiv a \operatorname{Im} B(x_1) + b \operatorname{Re}\{(Ax_1, x_2) - (Ax_2, x_1)\} = 0$, a system clearly possessing (two) solutions.

REMARKS. For background and (some) previous proofs, see [1]–[6]; our approach (used in a less simple way in [7], from which the demonstration above evolved) of continuity arguments utilizing conic sections is also present in [6] and is certainly not new as a general technique.

## REFERENCES

1. O. Toeplitz, *Das algebraische Analogon zu einem Satz von Fejér*, Math. Z. **2** (1918), 187–197.

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2. F. Hausdorff, *Der Wertvorrat einer Bilinearform*, Math. Z. **3** (1919), 314–316.

3. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloq. Publ., vol. 15, Amer. Math. Soc., Providence, R. I., 1932, pp. 130–133.

4. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967, pp. 317–318. MR **34** #8178.

5. W. F. Donoghue, *On the numerical range of a bounded operator*, Michigan Math. J. **4** (1957), 261–263. MR **20** #2622.

6. R. Raghavendran, *Toeplitz-Hausdorff theorem on numerical ranges*, Proc. Amer. Math. Soc. **20** (1969), 284–285. MR **38** #1509.

7. K. Gustafson, *A min-max theorem*, Notices Amer. Math. Soc. **15** (1968), 799. Abstract #68T-B38.

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