

SHORTER NOTES

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THE TOEPLITZ-HAUSDORFF THEOREM FOR LINEAR OPERATORS

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THEOREM (TOEPLITZ-HAUSDORFF). *The numerical range $W(A) = \{(Ax, x) : \|x\| = 1, x \in D(A)\}$ of an arbitrary (perhaps unbounded and not densely defined) linear operator A in a (pre)-Hilbert space (real or complex) is convex.*

PROOF. Since $W(\mu A + \gamma) = \mu W(A) + \gamma$, for scalars μ, γ , it suffices to consider the situation $(Ax_1, x_1) = 0, (Ax_2, x_2) = 1, \|x_i\| = 1, x_i \in D(A), i = 1, 2$. Let $x = \alpha x_1 + \beta x_2, \alpha$ and β real, and require

$$(1) \quad \|x\|^2 \equiv \alpha^2 + \beta^2 + 2\alpha\beta \operatorname{Re}(x_1, x_2) = 1,$$

and desire (for each $0 < \lambda < 1$)

$$(2) \quad (Ax, x) \equiv \beta^2 + \alpha\beta\{(Ax_1, x_2) + (Ax_2, x_1)\} = \lambda.$$

Let $B = (Ax_1, x_2) + (Ax_2, x_1)$; if B is real, then the system (1), (2) describes an ellipse (intercepts $\pm 1, \pm 1$) and a hyperbola (intercepts $\pm \lambda^{1/2}$) and clearly possesses (four, since $|\operatorname{Re}(x_1, x_2)| < 1$ by Schwarz's Inequality) solutions. But B can always be guaranteed real by using an appropriate (scalar multiple of) x_1 ; i.e., explicitly, use $x'_1 = \mu x_1$, where $\mu = a + ib$ satisfies (1') $|\mu|^2 \equiv a^2 + b^2 = 1$ and (2') $\operatorname{Im} B(x'_1) \equiv a \operatorname{Im} B(x_1) + b \operatorname{Re}\{(Ax_1, x_2) - (Ax_2, x_1)\} = 0$, a system clearly possessing (two) solutions.

REMARKS. For background and (some) previous proofs, see [1]–[6]; our approach (used in a less simple way in [7], from which the demonstration above evolved) of continuity arguments utilizing conic sections is also present in [6] and is certainly not new as a general technique.

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