Let $G$ be a Hausdorff topological group. We shall be dealing with positive and real valued Radon measures on $G$. Following [1], a real valued Radon measure $\mu$ on $G$ is said to be mobile if, for every compact set $K \subseteq G$, the function $\sigma \mapsto \mu(\sigma K)$ is continuous on $G$. In the case of a locally compact Hausdorff group any bounded mobile Radon measure is the indefinite integral (with respect to Haar measure) of a function in $L^1(G)$ [1].

We shall prove in this note that if there exists a (nontrivial) mobile real (not necessarily bounded) Radon measure on $G$, then the group is locally compact (Theorem 2). An immediate consequence is the following converse to the existence of Haar measure. If in a Hausdorff topological group there is a nontrivial ($\neq 0$) left translation invariant Radon measure then the group is locally compact.

Let us first recall

**Definition 1.** A Radon measure $\mu$ on a Hausdorff topological space $X$ is a positive measure defined on the Borel subsets of $X$ satisfying
1. $\mu$ is locally finite and 2. $\mu$ is inner regular i.e. for every Borel set $B$, $\mu(B) = \sup \{\mu(K) : \text{compact } K \subseteq B\}$. A real valued Radon measure $\nu$ is a signed Borel measure such that $\nu^+$ and $\nu^-$ are both Radon (equivalently $\nu$ is the difference of two positive Radon measures, one of which (at least) is totally finite).

We shall need the following fact about Radon measures. For every compact set $K$ and any $\epsilon > 0$, there exists an open set $V \supseteq K$ such that $\mu(V) < \mu(K) + \epsilon$.

**Lemma 1.** Let $\mu$ be a (nonnegative) mobile Radon measure on $G$. Then, for each compact subset $K$ of $G$, $\mu(K \triangle \sigma K)$ tends to zero as $\sigma$ tends to the identity element $e$.

**Proof.** Given $\epsilon > 0$, let $U$ be an open set containing $K$ such that $\mu(U) < \mu(K) + \epsilon/3$. Since $\mu$ is mobile, we can choose a neighbourhood $V$ of $e$ such that $|\mu(K) - \mu(\sigma K)| < \epsilon/3$ for every $\sigma \in V$ and further we can assume that $VK \subseteq U$. Then, $\mu(U - \sigma K) \leq 2\epsilon/3$ and we get

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\[ \mu(\sigma K \triangle K) \leq \mu(U - K) + \mu(U - \sigma K) < \epsilon \]

for every \( \sigma \in V \). The proof is complete.

**Theorem 1.** If \( \mu \) is a mobile real Radon measure on \( G \), then both \( \mu^+ \) and \( \mu^- \) are mobile.

**Proof.** It can be easily proved that \( \mu^+ \) would be mobile if for every compact set \( K \subset G \), the function \( \sigma \to \mu^+(\sigma K) \) is continuous at \( e \). We may also assume that \( \mu^+(G) < +\infty \). Let \( G = P \cup N \), with \( P \) a positive set for \( \mu \) and \( N \) a negative set for \( \mu \). (\( P \) and \( N \) can be chosen to be Borel sets.) Let \( \epsilon > 0 \). We choose a compact set \( E \subset P \) such that \( \mu(P) - \epsilon < \mu(E) \). Since \( \mu^-(E) = 0 \), we can choose \( U \) open \( \supseteq E \) satisfying \( \mu^-(U) < \epsilon \). Then, we also have \( \mu^+(U - E) < \epsilon \). Now, we select a neighbourhood \( V \) of \( e \) such that

(i) \( V \subset U \),
(ii) \( |\mu(\sigma E) - \mu(E)| < \epsilon \) for each \( \sigma \in V \), and
(iii) \( |\mu(K \cap E) - \mu[\sigma(K \cap E)]| < \epsilon \) for each \( \sigma \in V \).

Then

\[
|\mu(E) - \mu(\sigma E \cap (U - E)) - \mu[\sigma E \cap (U - E)]| < \epsilon,
\]

and so,

\[
|\mu(E) - \mu(\sigma E \cap E)| \leq |\mu|[\sigma E \cap (U - E)] + \epsilon
\]

\[
\leq |\mu|(U - E) + \epsilon < 3\epsilon.
\]

Also,

\[
\mu[\sigma(K \cap E)] = \mu(\sigma K \cap \sigma E)
\]

\[
= \mu(\sigma K \cap \sigma E \cap U) + \mu[\sigma K \cap \sigma E \cap (U - E)].
\]

Hence,

\[
|\mu(K \cap E) - \mu(\sigma K \cap \sigma E \cap E)| \leq |\mu|[(E \cap \sigma K) - \sigma E] + \epsilon < 3\epsilon.
\]

Further, since \( \mu \) is a positive measure on \( E \),

\[
0 \leq \mu(\sigma K \cap E) - \mu(\sigma K \cap \sigma E \cap E) = \mu[(E \cap \sigma K) - \sigma E] \leq \mu(E - \sigma E) = \mu(E) - \mu(E \cap \sigma E) < 3\epsilon \quad \text{from (1)}.
\]

Combining (2) and (3) we get,

\[
|\mu(K \cap E) - \mu(\sigma K \cap E)| < 6\epsilon.
\]
Now, for any \( \sigma \in V \),
\[
\left| \mu^+(\sigma K) - \mu^+(K) \right| = \left| \mu(\sigma K \cap E) - \mu(K \cap E) + \mu[\sigma K \cap (P - E)] - \mu[K \cap (P - E)] \right| \\
\leq \left| \mu(\sigma K \cap E) - \mu(K \cap E) \right| + \left| \mu[\sigma K \cap (P - E)] \right| + \left| \mu[K \cap (P - E)] \right| < 8\varepsilon.
\]

Since this is valid for every \( \varepsilon > 0 \), we conclude that \( \mu^+ \) is mobile. Also \( \mu^- = \mu^+-\mu \) is mobile and this completes the proof.

**Lemma 2.** Let \( \mu \) be a nontrivial Radon measure on \( G \). If there exist \( K \) compact \( \subset G \), a positive number \( \delta \) and a neighbourhood \( V \) of \( e \) satisfying (i) \( 0 < \delta < \mu(K) \), and (ii) \( \forall \sigma \in V, \mu(\sigma K \Delta K) < \delta \), then \( G \) is locally compact.

**Proof.** We observe that
\[
KK^{-1} \supset \{ \sigma : \sigma K \cap K \neq \emptyset \} \supset \{ \sigma : \mu(\sigma K \Delta K) < \delta \}.
\]
Hence, the compact set \( KK^{-1} \) contains the neighbourhood \( V \) of \( e \) (by hypothesis). It follows that \( G \) is locally compact completing the proof.

**Theorem 2.** Let \( G \) be a Hausdorff topological group and \( \mu \) a nontrivial real valued Radon measure on \( G \) which is mobile. Then \( G \) is locally compact.

**Proof.** Since \( \mu^+ \) and \( \mu^- \) are both mobile Radon measures (Theorem 1), we may assume that \( \mu^+ \) is nontrivial. Let \( K \) be a compact set with \( \mu^+(K) > 0 \) and \( 0 < \delta < \mu^+(K) \). Then a neighbourhood \( V \) of \( e \) could be chosen (Lemma 1) so that \( \mu^+, K, \delta \) and \( V \) satisfy the conditions of Lemma 2. The required result follows.

The following corollary is an immediate consequence.

**Corollary 1.** Let \( G \) be a Hausdorff topological group with a nontrivial left translation invariant Radon measure. Then \( G \) is locally compact.

**Corollary 2.** Let \( G \) be a topological group of which the underlying space is Souslin. If there exists a nontrivial locally finite mobile (i.e. \( \sigma \rightarrow \mu(\sigma K) \) continuous) Borel measure on \( G \), then \( G \) is locally compact.

This is a consequence of the fact [3, Chapter II] that every locally finite Borel measure on a Souslin space is Radon. This result includes as a particular case a result due to J. C. Oxtoby [2, Theorem 1] on polish groups.
Bibliography


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