

ON MAPPING INDECOMPOSABLE CONTINUA ONTO CERTAIN CHAINABLE INDECOMPOSABLE CONTINUA

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1. **Introduction.** The classical results on mappings (i.e. continuous transformations) involve arcs, either as the domain space (the Hahn-Mazurkiewicz theorem) or as the range space (Urysohn's characterization of normality). Not enough of whatever common structure exists among more complicated continua (compact connected metric spaces) has as yet been perceived to allow the formulation of many mapping theorems of a general nature. Among the known results is the fact that not every continuum is a continuous image of a plane continuum [5], while every continuum is an image of some one-dimensional continuum [8], [11] (and hence of a continuum in E^3). Studies of circle-like continua have also yielded interesting theorems, [3], [13].

Even indecomposable continua, generally considered highly pathological, have shown promise of having interesting mapping properties. The example of a chainable continuum which can be mapped onto every chainable continuum, given in [4], [9], and [12], in each case was the pseudo-arc, which is hereditarily indecomposable. A continuum with similar properties with respect to circularly chainable continua has been found, [13], and again is hereditarily indecomposable.

In this paper, we show in §2 that there is a chainable indecomposable continuum D which is a continuous image of every indecomposable continuum. In §3, we study some indecomposable continuous images of D (each of which is also, of course, an image of every indecomposable continuum).

It is hoped the techniques used here will prove useful in further studies of the mapping properties of indecomposable continua; for example, to settle the following questions.

- (1) Is the pseudo-arc a continuous image of every hereditarily indecomposable continuum?
- (2) Is every continuum a continuous image of some (hereditarily) indecomposable continuum?

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2. Mapping indecomposable continua onto D .

DEFINITIONS. By a *map* we mean a continuous transformation. A *polyhedron* is a nondegenerate finitely triangulable continuum. By an *inverse sequence* $\{P_i, f_i\}$ on polyhedra we mean a sequence such that for each positive integer i , P_i is a polyhedron and f_i maps P_{i+1} onto P_i . We use π_n to denote the projection map from $\lim\{P_i, f_i\}$ onto P_n . If $i < j$, $f_{ij} = f_i \cdot \cdot \cdot f_{j-1}$. Let $I = [0, 1]$.

Let $D = \lim\{I, g_i\}$, where for each i , $g_i(x) = 2x$ if $0 \leq x \leq \frac{1}{2}$, and $g_i(x) = 2 - 2x$ if $\frac{1}{2} \leq x \leq 1$. D is homeomorphic to a standard example of an indecomposable plane continuum, described in [1, Example 6, p. 662]. The next theorems show that any indecomposable continuum can be mapped onto D .

It is known that every continuum is the limit of an inverse sequence on polyhedra [6] (the result may also be obtained from [10]).

We suppose in Theorems 1-4 that M is an indecomposable continuum which is the limit of an inverse sequence $\{P_i, f_i\}$ on polyhedra.

THEOREM 1. *If $\delta > 0$, n is a positive integer, and A and B are points of P_n , then there are a positive integer $m > n$ and points A' and B' of P_m such that $d[A, f_{nm}(A')] < \delta$, $d[A, f_{nm}(B')] < \delta$, and if α is an arc in P_m from A' to B' , then $d[B, f_{nm}(\alpha)] < \delta$.*

PROOF. Suppose not. Then there are a number $\epsilon > 0$, a positive integer n , and points A and B of P_n such that if $m > n$ and A' and B' are points of P_m such that $d[A, f_{nm}(A')] < \delta$ and $d[A, f_{nm}(B')] < \delta$, then there is an arc α in P_m from A' to B' such that $d[B, f_{nm}(\alpha)] \geq \delta$. Let X and Y denote points in different composants of M such that $d[A, \pi_n(X)] < \delta$ and $d[A, \pi_n(Y)] < \delta$. Then there are an increasing sequence of positive integers $\{n_i\}$ and a sequence of arcs $\{\alpha_i\}$ such that for each i , α_i is an arc in P_{n_i} from $\pi_{n_i}(X)$ to $\pi_{n_i}(Y)$ and $d[B, f_{n_i}(\alpha_i)] \geq \delta$. The limiting set L of the sequence $\pi_{n_1}^{-1}(\alpha_1), \pi_{n_2}^{-1}(\alpha_2), \cdot \cdot \cdot$ of subsets of M is a subcontinuum of M that contains both X and Y , but no point of $\pi_n^{-1}(B)$, contrary to the fact that M is irreducible from X to Y .

THEOREM 2. *If $\epsilon > 0$, n is a positive integer, and A and B are points of P_n , then there are (1) an integer $m > n$, (2) points A' and B' of P_m , and (3) a map ϕ from P_n onto P_n such that*

- (a) $d(\phi, 1) < \epsilon$ (where 1 denotes the identity on P_n),
- (b) $A = \phi f_{nm}(A') = \phi f_{nm}(B')$, and
- (c) $[\phi f_{nm}]^{-1}(B)$ separates A' from B' in P_m .

PROOF. There exist (1) connected open sets R_A and R'_A , containing A , of diameter $< \epsilon$, such that \bar{R}'_A lies in R_A , (2) similar sets R_B and R'_B for B , and (3) a map ϕ from P_n onto P_n such that

- (a) \bar{R}_A and \bar{R}_B are mutually exclusive,
- (b) ϕ is the identity on $P_n - (R_A + R_B)$,
- (c) $\phi(R'_A) = A$ and $\phi(R'_B) = B$, and
- (d) $\phi(R_A) = R_A$ and $\phi(R_B) = R_B$. Clearly $d(\phi, 1) < \epsilon$.

Let δ denote a positive number less than $d(A, P_n - R'_A)$ and $d(B, P_n - R'_B)$, and let m, A' , and B' be as given in Theorem 1 with respect to δ, n, A , and B . Since, from Theorem 1, $d[A, f_{nm}(A')] < \delta$, $f_{nm}(A')$ lies in R'_A , so $\phi f_{nm}(A') = A$. Similarly, $\phi f_{nm}(B') = B$. Now, if α is an arc in P_m from A to B , then there is a point Y of α such that $d[B, f_{nm}(Y)] < \delta$, and so $\phi f_{nm}(Y) = B$. Thus, since every arc in P_m from A' to B' intersects $[\phi f_{nm}]^{-1}(B)$, that set separates A' from B' in P_m .

THEOREM 3. *There are (1) an inverse sequence $\{P'_i, g_i\}$ on polyhedra and (2) sequences $\{A_i\}$ and $\{B_i\}$ such that for each i ,*

- (a) A_i and B_i are points of P_i ,
- (b) $A_i = g_i(A_{i+1}) = g_i(B_{i+1})$,
- (c) $g_i^{-1}(B_i)$ separates A_{i+1} from B_{i+1} in P_{i+1} , and
- (d) *there is a map that throws M onto $\lim \{P_i, g_i\}$.*

PROOF. We define $\{P'_i, g_i\}$ inductively from the sequence $\{P_i, f_i\}$ for M . Let $P'_1 = P_1$, and let A_1 and B_1 denote points of P'_1 . For each n , let $b_n = 2^{-n}$ and let $a_1 = m_1 = 1$. Let m_2, A_2, B_2 , and ϕ_1 satisfy the conditions of Theorem 4 for m, A', B' , and ϕ , respectively, regarding both ϵ and n in that theorem as 1, and A and B as A_1 and B_1 , respectively. Let $g_1 = \phi_1 f_{m_1 m_2}$ and $P'_2 = P_{m_2}$.

In general, given, for $j > 1$, $g_{j-1}, m_j, P'_j = P_{m_j}, A_j$, and B_j , let a_j denote a positive number such that if $i < j$ and X and Y are points of P'_j , and $d(X, Y) < a_j$, then $d[g_{ij}(X), g_{ij}(Y)] < b_j$. Let $m_{j+1}, A_{j+1}, B_{j+1}$, and ϕ_j satisfy the conditions of Theorem 2 for m, A', B' , and ϕ , respectively, regarding ϵ as a_j , n as m_j , A as A_j and B as B_j . Let $g_j = \phi_j f_{m_j m_{j+1}}$ and $P'_{j+1} = P_{m_{j+1}}$.

Then $\{P'_i, g_i\}$ is an inverse sequence, and the sequences $\{A_i\}$ and $\{B_i\}$ satisfy requirements (a)-(c) of this theorem. For each i , let $f'_i = f_{m_i m_{i+1}}$. Then for each i , $d(f'_i, g_i) < a_i$. Hence if (1) $S = \lim \{P'_i, f'_i\}$, which is homeomorphic to M , and (2) $T = \lim \{P'_i, g_i\}$, then the conditions of M. Brown's Theorem 1 [2, p. 479] are satisfied, for the sequence $\{a_i\}$ defined above is a Lebesgue sequence for $\{P'_i, g_i\}$, in Brown's terminology. (The reader should note that Brown's notation for inverse limits differs from the author's. In particular Brown's re-

quirement “ $\|f_{i+1} - g_{i+1}\| < a_i$ ” in the notation of this paper would be “ $\|f_i - g_i\| < a_i$.”) Hence there is a map from S onto T , and so M can be mapped onto T .

THEOREM 4. *M can be mapped onto D . Hence D is a chainable indecomposable continuum which is a continuous image of every indecomposable continuum.*

PROOF. Let $\{P'_i, g_i\}$, $\{A_i\}$, and $\{B_i\}$ denote sequences as defined in Theorem 3 such that M can be mapped onto $\lim \{P'_i, g_i\}$. We show that $\lim \{P'_i, g_i\}$ can be mapped onto D .

For each n , let α_{n+1} and β_{n+1} denote two mutually separated subsets of P_{n+1} such that A_{n+1} lies in α_{n+1} , B_{n+1} lies in β_{n+1} , and $\alpha_{n+1} + \beta_{n+1} = P'_{n+1} - g_n^{-1}(B_n)$. Let $\{\phi_i\}$ denote a sequence of maps such that (1) ϕ_1 maps P'_1 onto $[0, 1]$ and $\phi_1(A_1) = 0$ and $\phi_1(B_1) = 1$, and (2) if $n \geq 1$, and X is a point of P'_{n+1} , then (1) $\phi_{n+1}(X) = \frac{1}{2}$ if X is in $g_n^{-1}(B_n)$, (2) $\phi_{n+1}(X) = \frac{1}{2}\phi_n g_n(X)$ if X is in α_{n+1} , and (3) $\phi_{n+1}(X) = 1 - \frac{1}{2}\phi_n g_n(X)$ if X is in β_{n+1} .

By definition, $\phi_n(A_n) = 0$ and $\phi_n(B_n) = 1$ if $n = 1$; and if it is true when $n = k$, then

$$\begin{aligned} \phi_{k+1}(A_{k+1}) &= \frac{1}{2}\phi_k g_k(A_{k+1}) = \frac{1}{2}\phi_k(A_k) = 0, & \text{and} \\ \phi_{k+1}(B_{k+1}) &= 1 - \frac{1}{2}\phi_k g_k(B_{k+1}) = 1 - \frac{1}{2}\phi_k(A_k) = 1. \end{aligned}$$

Also for each n , $\phi_n g_n = w_2 \phi_{n+1}$. For suppose X is a point of P_{n+1} . If X is in $g_n^{-1}(B_n)$, then

$$\phi_n g_n(X) = \phi_n(B_n) = 1 = w_2(\frac{1}{2}) = w_2 \phi_{n+1}(X).$$

If X is in α_{n+1} , then

$$w_2 \phi_{n+1}(X) = 2[\frac{1}{2}\phi_n g_n(X)] = \phi_n g_n(X).$$

If X is in β_{n+1} , then

$$w_2 \phi_{n+1}(X) = 2 - 2[1 - \frac{1}{2}\phi_n g_n(X)] = \phi_n g_n(X).$$

Hence the sequence $\{\phi_i\}$ induces a mapping from $\lim \{P_i, g_i\}$ onto D .

3. Some indecomposable images of D . For each $n > 1$, let w_n denote the map from I onto I such that (1) if $0 \leq i \leq n$, then $w_n(i/n)$ is 0 if i is even and 1 if i is odd, and (2) w_n is linear on the interval $[i/n, (i+1)/n]$ for each i ($0 \leq i < n$). Let V denote the collection of all continua homeomorphic to the limit of an inverse sequence $\{I, f_i\}$ where for each i , $f_i = w_n$ for some positive integer n . The continuum D is an element of V since $D = \lim \{I, w_2\}$.

In this section we show (Theorem 8) that V contains every con-

tinuum homeomorphic to the limit of an inverse sequence $\{I, f_i\}$, where for each i , f_i is a nonmonotone near-open map from I onto I (a *near-open* map is a uniform limit of open maps), and further that each element of V can be mapped onto every element of V (Theorem 9). Hence each element of V is a continuous image of D , and therefore of every indecomposable continuum.

It would be of interest to know whether D can be mapped onto every chainable indecomposable continuum with arcwise connected composants. Not every such continuum is an element of V since there is one with three endpoints [7, pp. 141–142], while each element of V has fewer than three. The dyadic solenoid is a nonchainable indecomposable continuum with arcwise connected composants which is not a continuous image of D [5].

The proof of the following lemma is elementary.

LEMMA 1. *A map f from I onto I is open if and only if there is a sequence $0 = a_0 < a_1 < \dots < a_n = 1$ such that for each positive integer i , f restricted to $[a_{i-1}, a_i]$ is a homeomorphism from $[a_{i-1}, a_i]$ onto I .*

We call the integer n associated with f in the lemma the *order* of the open map f . Then for each n , w_n is an open map from I onto I of order n .

LEMMA 2. *Suppose α is a map from I onto I and a_0, \dots, a_n is an increasing sequence of numbers in I such that $\alpha(a_i) = 0$ if i is even and $\alpha(a_i) = 1$ if i is odd. Then there is a map β from I onto I such that $\alpha = w_n\beta$. Moreover, if $a_0 = 0$, $a_n = 1$, and for each i , $\alpha|_{[a_{i-1}, a_i]}$ is a homeomorphism from $[a_{i-1}, a_i]$ onto I , then β is a homeomorphism from I onto I such that $\beta(0) = 0$ and $\beta(1) = 1$.*

PROOF. Let $b_0 = 0$, $b_1 = 1$ and if $0 < i < n$, let $b_i = a_i$. Let β denote the function such that if $b_i \leq x \leq b_{i+1}$, then $\beta(x) = [i + \alpha(x)]/n$ if i is even, and $\beta(x) = [(i+1) - \alpha(x)]/n$ if i is odd. Clearly β is continuous, and straightforward calculations show that (1) $\alpha = w_n\beta$, (2) $\beta(a_0) = 0$, and (3) $\beta(a_n) = 1$. Hence β maps I onto I .

DEFINITION. If f maps I onto I , let $f^* = 1 - f$.

THEOREM 5. *Suppose $H = \lim \{I, f_i\}$ and $K = \lim \{I, g_i\}$, where for each i , f_i is an open map of order n_i from I onto I , and g_i is (1) w_{n_i} if $f_i(0) = 0$, and (2) $w_{n_i}^*$ if $f_i(0) = 1$. Then H and K are homeomorphic.*

PROOF. We define a sequence $\{\beta_i\}$ of homeomorphisms from I onto I such that for each i , $\beta_i(0) = 0$, $\beta_i(1) = 1$, and $\beta_i f_i = g_i \beta_{i+1}$. Let β_1 denote the identity on I , and suppose β_1, \dots, β_k have been appropriately defined. Let $0 = a_0 < a_1 < \dots < a_{n_k} = 1$ denote the increasing sequence of numbers associated with the open map f_k as in Lemma 1.

If $f_k(0) = 0$, then this sequence satisfies the requirements of Lemma 2, considering α as $\beta_k f_k$, and there is a homeomorphism β_{k+1} from I onto I such that $\beta_{k+1}(0) = 0$, $\beta_{k+1}(1) = 1$, and $\beta_k f_k = w_{n_k} \beta_{k+1} = g_k \beta_{k+1}$. If $f_k(0) = 1$, then this sequence satisfies the requirements of Lemma 2, considering α as $1 - \beta_k f_k$, and there is a homeomorphism β_{k+1} from I onto I such that $\beta_{k+1}(0) = 0$, $\beta_{k+1}(1) = 1$, and $1 - \beta_k f_k = w_{n_k} \beta_{k+1}$. Hence $\beta_k f_k = w_{n_k}^* \beta_{k+1} = g_k \beta_{k+1}$.

The sequence $\{\beta_i\}$ induces a homeomorphism from H onto K .

DEFINITION. If k is a positive integer such that if $1 \leq i \leq k$, then n_i is a positive integer greater than 1 and g_i is either w_{n_i} or $w_{n_i}^*$, and $v = g_1 \circ g_2 \circ \dots \circ g_k$, then let $|v|$ denote $w_{n_1} \circ w_{n_2} \circ \dots \circ w_{n_k}$. Let the collection W consist of w_2, w_3, \dots , and the collection W^* consist of $w_2, w_2^*, w_3, w_3^*, \dots$.

It can be verified that $w_n w_m = w_{nm}$, so that finite composites of elements of W are in W . Furthermore, if f maps I onto I , then if n is even, $w_n f^* = w_n f$ and $w_n^* f = w_n^* f^* = (w_n f)^*$, while if n is odd, $w_n^* f^* = w_n f$ and $w_n^* f = w_n f^* = (w_n f)^*$. It follows that if v_1 and v_2 are in W^* , then $v_1 v_2$ is either $|v_1 v_2|$ or $|v_1 v_2|^*$, so that finite composites of elements of W^* are in W^* .

The next theorem roughly states that any number of maps in an inverse sequence involving w_2, w_3, \dots may be "flipped over" without affecting the limit. This is not true in general, for let $f(0) = 0, f(\frac{1}{2}) = 1, f(1) = \frac{1}{2}$, and f be linear on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$. Then $\lim \{I, f\}$ is a $\sin(1/x)$ -continuum, while $\lim \{I, f^*\}$ is indecomposable.

THEOREM 6. *If, for each i, n_i is a positive integer and g_i is either w_{n_i} or $w_{n_i}^*$, then $\lim \{I, g_i\}$ and $\lim \{I, |g_i|\} = \lim \{I, w_{n_i}\}$ are homeomorphic.*

Before proving Theorem 6, we prove a lemma.

LEMMA 3. *If for each i, n_i is an even positive integer, then $H = \lim \{I, w_{n_i}^*\}$ and $K = \lim \{I, w_{n_i}\}$ are homeomorphic.*

PROOF. For each i , let β_i denote the map from I onto I such that $\beta_i(x) = 1 - x$. Then, since n_i is even, $\beta_i w_{n_i}^* = w_{n_i} = w_{n_i} \beta_{i+1}$. Hence the sequence $\{\beta_i\}$ induces a homeomorphism from H onto K .

PROOF OF THEOREM 6. We first suppose that for each i, k_i is an even positive integer, and that $\{j_i\}$ is an infinite increasing sequence of positive integers such that $g_{j_i} = w_{k_i}$. Then for each i , let $v_i = g_{j_i+1, j_i+1}$. Since k_i is even, $w_{k_i} v_i = w_{k_i} |v_i|$ so that $\lim \{I, w_{k_i} v_i\}$ is homeomorphic to $\lim \{I, w_{k_i} |v_i|\}$. But the former is homeomorphic to $\lim \{I, g_i\}$, while the latter is homeomorphic to $\lim \{I, |g_i|\}$. Under this supposition, then, the theorem is true.

We now suppose that for each i, k_i is an even positive integer, and that $\{j_i\}$ is an infinite increasing sequence of positive integers such

that $g_{j_i} = w_{k_i}^*$. Again for each i , let $v_i = g_{j_{i+1}, j_{i+1}}$. But $w_{k_i}^* v_i = (w_{k_i} | v_i |)^*$, and $w_{k_i} | v_i |$ is an element of W of even order. Hence by Lemma 3, $\lim \{I, w_{k_i}^* v_i\}$ is homeomorphic to $\lim \{I, w_{k_i} | v_i |\}$. But the former is homeomorphic to $\lim \{I, g_i\}$ and the latter to $\lim \{I, |g_i|\}$.

The only case left is that in which all but finitely many of the integers $\{n_i\}$ are odd, and since for each n , $\lim \{I, g_i\}$ and $\lim \{I, g_{i+n}\}$ are homeomorphic, we assume that all the integers $\{n_i\}$ are odd. Let $\{\beta_i\}$ denote a sequence of maps from I onto I such that (1) β_1 is the identity on I , and (2) $\beta_{j+1} = \beta_j$ if $g_j = w_{n_j}$, and $\beta_{j+1} = \beta_j^*$ if $g_j = w_{n_j}^*$. Then $\beta_j g_j = w_{n_j} \beta_{j+1}$, and $\{\beta_i\}$ induces a homeomorphism from $\lim \{I, g_i\}$ onto $\lim \{I, |g_i|\}$.

THEOREM 7. *If, for each i , f_i is an open map from I onto I of order $n_i > 1$, then $\lim \{I, f_i\}$ and $\lim \{I, w_{n_i}\}$ are homeomorphic.*

PROOF. This follows directly from Theorems 5 and 6.

THEOREM 8. *If, for each i , f_i is a nonmonotone near-open map from I onto I , then $\lim \{I, f_i\}$ is an element of V .*

PROOF. This follows from Theorem 7 and Brown's approximation theorem [2, Theorem 3, p. 481], since each nonmonotone near-open map is the uniform limit of open maps of order > 1 .

THEOREM 9. *If M and N are elements of V , then there is a map from M onto N .*

PROOF. M is the limit of an inverse sequence $\{I, f_i\}$ where for each i , f_i is in W . There is a similar sequence $\{I, g_i\}$ for N . For each i , let l_i denote the order of g_i . Let $\{n_i\}$ denote an increasing sequence of positive integers such that for each i , the order of $f_{n_i n_{i+1}}$ is greater than l_i , and for each i , let $f'_i = f_{n_i n_{i+1}}$. Then $\lim \{I, f'_i\}$ is homeomorphic to M . There is a sequence $\{\beta_i\}$ of maps from I onto I such that β_1 is the identity on I and for each n , $\beta_n f'_n = g_n \beta_{n+1}$; for if β_n is defined, it is easily seen that there is an increasing sequence a_0, \dots, a_{l_n} of numbers in I such that $\beta_n f'_n(a_i)$ is 0 if i is even, and 1 if i is odd, and so Lemma 2 gives a map β_{n+1} such that $\beta_n f'_n = g_n \beta_{n+1}$. The sequence $\{\beta_i\}$ induces a map from $\lim \{I, f'_i\}$ onto N . So M can be mapped onto N .

COROLLARY. *Each element of V is an indecomposable chainable continuum which is a continuous image of every indecomposable continuum.*

The referee has suggested that it would be of interest to know whether V contains three topologically different continua. There are two, since $\lim \{I, w_2\}$ has only one endpoint and $\lim \{I, w_3\}$ has two.

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