TAME BOUNDARY SETS OF CRUMPLED CUBES IN $E^3$  

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Abstract. If a crumpled cube $K$ in $E^3$ is re-embedded by a homeomorphism $h$ such that $h(K)$ is tame from $\text{Ext } h(K)$ and $F$ is a tame closed subset of $\text{Bd } K$ which either has no degenerate components or consists entirely of degenerate components, then $h(F)$ is tame.

Hosay [4] and Lininger [5] have independently shown that any crumpled cube in $E^3$ may be re-embedded in $E^3$ so that it is tame from its exterior. The question arises as to what properties of subsets of the boundary of the crumpled cube remain invariant under such a re-embedding. In particular if $F$ is a tame closed subset of the boundary of a crumpled cube $K$ in $E^3$ and $h$ is a homeomorphism of $K$ into $E^3$ such that $h(K)$ is tame from $\text{Ext } h(K)$, is $h(F)$ tame? If $F$ has no degenerate components, then it follows from a recent result of Cannon [2] along with results of Lister [6] and Loveland [7] that $h(F)$ is tame. We show below that $h(F)$ is tame in case $F$ consists entirely of degenerate components. For an arbitrary closed set $F$ the question is open.

Theorem. If $K$ is a crumpled cube in $E^3$, $F$ is a tame closed 0-dimensional subset of $\text{Bd } K$, and $h$ is a homeomorphism of $K$ into $E^3$ such that $h(K)$ is tame from $\text{Ext } h(K)$, then $h(F)$ is tame.

Proof. By Bing's characterization of tame 0-dimensional sets in $E^3$ [1] there is a finite collection $C_1$ of mutually exclusive simple neighborhoods $\{N_{11}, N_{12}, \ldots, N_{1n(1)} \}$ of mesh less than 1 covering $F$. Let $\varepsilon_1 = (1/4)D_1$ where $D_1 = \rho(F, E^3 - \bigcup_{i=1}^{n(1)} N_{1i})$ and using a lemma of Daverman [3] let $h_1$ be an $\varepsilon_1$-homeomorphism of $K$ into $E^3$ and $S_1$ be a polyhedral 2-sphere homeomorphically within $\varepsilon_1$ of $\text{Bd } h_1(K)$ such that $h_1(K) \subseteq \text{Int } S_1$. Then $C_1$ covers $h_1(F)$, $\rho(h_1(p), p) < \varepsilon_1$, for each $p \in F$, and it is clear from the method Daverman uses in the proof of his lemma that $h_1(F)$ is tame.

Now there is a finite collection $C_2$ of mutually exclusive simple neighborhoods $\{N_{21}, N_{22}, \ldots, N_{2n(2)} \}$ of mesh less than $1/2$ such that $C_2$ covers $h_1(F)$ and each element of $C_2$ is a subset of an element of $C_1$. Let $\varepsilon_2 = (1/4)D_2$ where $D_2 = \min \{\varepsilon_1, \rho(h_1(F), E^3 - \bigcup_{i=1}^{n(2)} N_{2i})\}$.

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and let $h'_2$ be an $\varepsilon_2$-homeomorphism of $h_1(K)$ into $E^3$ and $S_2$ be a polyhedral 2-sphere homeomorphically within $\varepsilon_2$ of $\text{Bd } h'_2(h_1(K))$ such that $h'_2(h_1(K)) \subseteq \text{Int } S_2$. Then let $h_2 = h'_2 h_1$ and note that for each $p \in F$, $\rho(h_2(p), h_1(p)) < \varepsilon_2$, $\rho(h_2(p), \bar{p}) < \varepsilon_1 + \varepsilon_2 \leq \varepsilon_1 + (1/4) \varepsilon_1 \leq (1/4)D_1 + (1/16)D_1$, and $h_2(F)$ is tame.

Proceeding inductively we obtain

1. a nested sequence of collections $C_1, C_2, \ldots, C_m, \ldots$ of mutually exclusive simple neighborhoods such that $C_m = \{ N_{m_1}, N_{m_2}, \ldots, N_{m(m)} \}$ is of mesh less than $1/2^{m-1}$ and covers $h_i(F)$ for $0 < i < m$,

2. a sequence of homeomorphisms $h_1, h_2, \ldots, h_m, \ldots$ of $K$ into $E^3$ where $h_m = h_m' h_{m-1}$, $h_m'$ is an $\varepsilon_m$-homeomorphism of $h_{m-1}(K)$ into $E^3$, $\varepsilon_m = (1/4)D_m$, and $D_m = \min \{ \varepsilon_{m-1}, \rho(h_{m-1}(F), E^3 - \bigcup_{i=1}^{m-1} N_{m_i}) \}$, and

3. a sequence of polyhedral 2-spheres $S_1, S_2, \ldots, S_m, \ldots$ where $S_m$ is homeomorphically within $\varepsilon_m$ of $\text{Bd } h_m(K)$ and such that $h_m(K) \subseteq \text{Int } S_m$.

Note that for each $p \in F$, for $i > m$,

$$\rho(h_i(p), h_m(p)) < (1/4 + 1/16 + \cdots)D_m = (1/3)D_m.$$

Following Lininger [5] we obtain a limiting homeomorphism $h'$ of the sequence $h_1, h_2, \ldots, h_m, \ldots$ such that $h'(K)$ is tame from $\text{Ext } h'(K)$. Then from the last inequality we have for any positive integer $m$, $h'(F) \subseteq \bigcup_{i=1}^{m} N_{m_i}$. Hence $h'(F)$ is tame and it follows easily that $h(F)$ is tame.

References


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