

ON EXTENDING HOMEOMORPHISMS TO FRÉCHET MANIFOLDS¹

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ABSTRACT. Let M be a Fréchet manifold and K be a Z -set in M . It is shown that a homeomorphism h of K into M can be isotopically extended to a homeomorphism of M onto M if and only if $h(K)$ is a Z -set and h is homotopic to the identity in M . Conditions under which the isotopic extension can be required to be "close to" the homotopy are also given.

1. Introduction. A *Fréchet manifold* (F -manifold) is a separable metric space admitting an open cover by sets homeomorphic to separable Hilbert space l_2 or, equivalently [1], to s , the countable infinite product of open intervals $(-1, 1)$. It is well known that any F -manifold is topologically complete; i.e. can be assigned an equivalent complete metric.

CONVENTION. Throughout this paper we shall use X and Y to denote topologically complete metric spaces, I to denote the closed interval $[0, 1]$, and M to denote an F -manifold.

Recently, the topology of F -manifolds has been studied by various authors. We use several recent results in our arguments below. In this paper we answer a number of questions posed at a conference on infinite-dimensional topology in January, 1969, at Ithaca. Among other results, we establish (Theorem 4.2) necessary and sufficient conditions on isotopically extending homeomorphisms of certain closed subsets of an F -manifold to a homeomorphism of the manifold. We also investigate questions concerning the extensions of homeomorphisms or the replacement of homotopies by isotopies where the new maps must be "close to" the original ones.

Two related concepts allow us to measure such closeness. Let \mathfrak{u} be an open cover of a space Y . Maps f and g of a space X into Y are \mathfrak{u} -close if for each $x \in X$ there exists $U \in \mathfrak{u}$ such that $f(x) \in U$ and $g(x) \in U$. A homotopy $F: X \times I \rightarrow Y$ is *limited by* \mathfrak{u} if for each $x \in X$, $F(\{x\} \times I)$ is contained in some member of \mathfrak{u} . Observe that under a homeomorphism between spaces X and Y , a cover on X or Y induces a cover on the other. Thus under a space homeomorphism which need

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not be uniformly continuous, the concepts of “ \mathfrak{u} -close” and “limited by \mathfrak{u} ” may be used in lieu of possibly inapplicable ϵ -conditions.

A basic notion in the topology of F -manifolds is the notion of Z -sets, originally introduced as Property Z in [3]. A subset K in an F -manifold M is a Z -set if K is closed and if for every nonnull homotopically trivial open set U of M , $U \setminus K$ is nonnull and homotopically trivial. (A set U in a space is homotopically trivial if each map of the boundary of any n -ball into U can be extended to a map of the n -ball into U .) It is well known that any closed set which is a countable union of Z -sets is a Z -set. In [3] Z -sets are characterized in locally convex linear spaces homeomorphic to s . In (D) below we cite a similar characterization for Z -sets in F -manifolds due to Chapman [6].

Basic results concerning F -manifolds which we use are the following.

(A) The factor theorem (Anderson and Schori [5]). Every F -manifold M is homeomorphic to $M \times s$.

(B) The open embedding theorem (Henderson [7]). Every F -manifold can be embedded as an open subset of s .

The following theorems are proved using (A).

(C) The isotopy theorem (West [9]). For any homotopy F of $X \times I$ into M , there exists a pseudo-isotopy G between F_0 and F_1 such that $G|X \times (0, 1)$ is an embedding into M , and for each $t_0 \in (0, 1/2)$, $G(X \times [t_0, 1 - t_0])$ is a Z -set in M . Furthermore, for any open cover \mathfrak{u} of M , G may be taken to be \mathfrak{u} -close to F .

Before stating Chapman's characterization of Z -sets in F -manifolds, we need some further notation and definitions. Let π_A denote the projection map of $A \times B$ onto A . A closed set K has infinite deficiency in s if s can be written as a product $s(1) \times s(2)$ of two copies of itself with $\pi_{s(1)}(K)$ consisting of a single point. For any F -manifold M , a closed set K in $M \times s$ has infinite deficiency in $M \times s$ if $\pi_s(K)$ has infinite deficiency in s . Observe that if K has infinite deficiency in $M \times s$ it follows from (A) that there is a homeomorphism σ of $M \times s$ onto itself such that $\pi_s \circ \sigma(K)$ consists of a single point which may be taken to be the origin of s .

(D) The characterization of Z -sets (Chapman [6]). For any closed set K of M there is a homeomorphism σ of M onto $M \times s$ such that $\sigma(K)$ has infinite deficiency in $M \times s$ if and only if K is a Z -set.

2. Preliminary lemmas. We present here four lemmas which will be of use in the sequel. Lemma 2.1 is well known. Lemma 2.2 follows routinely from Lemma 2.1 by using a locally finite refinement of \mathfrak{u} employing box-like neighborhoods of s .

LEMMA 2.1. *Let X be a closed subset of M and let f be a real valued function defined on X such that f is locally bounded above zero. Then there exists a continuous real valued function g on M such that $g \geq 0$ and, on X , $0 < g \leq f$.*

LEMMA 2.2. *Let X be a closed subset of M and let \mathfrak{U} be an open cover of $X \times \{0\}$ in $M \times s$. Then there exists a sequence (ψ_i) of continuous real valued functions on M such that for each $i > 0$*

- (a) $0 \leq \psi_i$,
- (b) ψ_i is positive on X , and
- (c) for each $x \in M$ either $\psi_i(x) = 0$ for all i , or there exists $U \in \mathfrak{U}$ such that for each $y = (y_i) \in s$ satisfying $0 \leq |y_i| \leq \psi_i(x)$, $(x, y) \in U$.

Note that we may consider (ψ_i) to define a continuous function ψ of M into s .

The following lemma is a corollary of the apparatus of [4]. For a proof we need merely invoke (B) regarding M as openly embedded in s and then create infinite deficiency locally on $M \setminus K$.

LEMMA 2.3. *For any Z -set K in M and any open cover \mathfrak{U} of M , there is a homeomorphism h of M into M such that*

- (1) $h|K = \text{id}$,
- (2) $h(M)$ is a Z -set in M , and
- (3) h is \mathfrak{U} -close to the identity.

We shall use this lemma in replacing continuous functions into M by continuous functions into Z -sets of M which are “close to” the original ones. For simplicity, we shall omit the explicit use of the homeomorphism h when applying the lemma. We shall need the following lemma which is a slightly stronger version of (C). Its proof is based on (C) and (D).

LEMMA 2.4. *Let \mathfrak{U} be an open cover of M and let $G: X \times I \rightarrow M$ be a homotopy such that $G_0(X) \cup G_1(X)$ is a Z -set in M . Then there exists an isotopy $H: X \times I \rightarrow M$ such that $G_0 = H_0$, $G_1 = H_1$, $H(X \times I)$ is in a Z -set in M , H is \mathfrak{U} -close to G , and $H_{1/2}(X) \cap (H_0(X) \cup H_1(X)) = \square$.*

PROOF. By passing to a star-refinement of \mathfrak{U} we may assume from Lemma 2.3 that $G(X \times I)$ is contained in a Z -set homeomorphic to M . Then by (C) there exists such an isotopy but without the last condition. Call such an isotopy H^1 . By (D) there exists a homeomorphism f of M onto $M \times s$ such that $\pi_s \circ f \circ H^1(X \times I)$ consists of the origin of s . Let $I_i^0 = (-1, 1)$ be the i th factor of s and write $M \times s$ as $M \times I_1^0 \times (\pi_{j>i} I_j^0)$. By Lemma 2.2 there exists a continuous real valued function $\psi_1 > 0$ defined on $\pi_M \circ f \circ H^1(X \times I)$ and for any t with $|t| \leq 1$

there is some $U \in \mathfrak{U}$ such that $(y, 0, 0)$ and $(y, t \cdot \psi_1(y), 0)$ are in $f(U)$. Let H^2 be defined by

$$H^2(x, t) = (\pi_M \circ f \circ H^1(x, t), t \cdot \psi_1(\pi_M \circ f \circ H^1(x, t)), 0).$$

Then $H = f^{-1} \circ H^2$ is the desired isotopy.

The following lemma follows from a well-known theorem of Kuratowski and the fact that all separable infinite-dimensional Banach spaces are homeomorphic to s .

LEMMA 2.5. *If X is a complete separable metric space, then X may be embedded as a closed subset of s . Moreover, an embedding ν can be chosen such that $\pi_i \circ \nu(x) > 1$ for each x in X .*

PROOF. By the theorem of Kuratowski X may be embedded isometrically in a separable Banach space which is homeomorphic to s . The second part follows from the fact [2] that the countable product of half open intervals $[\frac{1}{2}, 0)$ is homeomorphic to s .

3. A mapping replacement theorem. Our first theorem gives a sufficient condition under which a homeomorphism of a closed subset A of a complete separable metric space X into an F -manifold M can be extended to a homeomorphism of X into M . In this theorem the condition that $f(A)$ be a Z -set cannot be omitted (even if the Z -set condition of the conclusion is dropped). For example, $f(A)$ might separate M in such a manner that $f(A)$ would separate $h(X)$ for any continuous extension h of $f|A$, whereas A might not separate X .

THEOREM 3.1. *Let A be a closed subset of X and f a continuous function of X into M such that $f|A$ is a homeomorphism of A onto a Z -set of M . Then there exists a homeomorphism h of X onto a Z -set in M such that $f|A = h|A$. Moreover, if \mathfrak{U} is some open cover of M , h may be taken \mathfrak{U} -close to f .*

PROOF. By Lemma 2.3 we may assume that $f(X)$ lies in a Z -set. From (D) there is a homeomorphism σ of M onto $M \times s$ such that $\pi_s \circ \sigma \circ f(X) = \{0\}$. Write s as a product $s(1) \times s(2)$ of two copies of itself and let ν be an embedding of X as a closed set in s such that $\nu(X) \geq 0$ (i.e. for $x \in X$, each coordinate of $\nu(x)$ is nonnegative) and $\pi_{s(2)} \circ \nu(X) = \{0\}$. Then the map g of X into $M \times s$ defined by $g(x) = (\pi_M \circ \sigma \circ f(x), \nu(x))$ is an embedding of X in $M \times s$ as a closed set of infinite deficiency. The projection π of $M \times s$ onto $M \times \{0\}$ induces a homeomorphism of $g(A)$ onto $\sigma \circ f(A)$. Thus by the method of Klee [8] as used in [2], $\pi|g(A)$ may be extended to a homeomorphism k of $M \times s$ onto itself such that $\pi_{s(2)} \circ k = \pi_{s(2)}$. Note that

$\sigma \circ k \circ \sigma^{-1}$ satisfies the theorem except for the closeness condition. Let \mathcal{U} denote the open cover of $M \times s$ induced by \mathcal{U} and σ . It suffices to obtain a homeomorphism of X onto a Z -set of $M \times s$ which extends $\sigma \circ f|A$ and is \mathcal{U} -close to $\sigma \circ f$.

By Lemma 2.2 there is a sequence of real valued continuous functions (ψ_i) on M satisfying (a) $\psi_i > 0$, and for each $x \in M$ there exists $V \in \mathcal{U}$ such that for $y = (y_i) \in s$ satisfying $0 \leq |y_i| \leq \psi_i(x)$, $(x, y) \in V$. Define h^1 of $M \times s$ onto $M \times s$ by $h^1(x) = (\pi_M \circ k(x), \psi(\pi_M \circ k(x)) \cdot \pi_s \circ k(x))$. (Here \cdot denotes coordinatewise multiplication, ψ is as in the remark after Lemma 2.2.) The homeomorphism h of X into M defined by $h = \sigma^{-1} \circ h^1 \circ g$ is the desired homeomorphism.

4. Replacing homotopies by ambient isotopies. The following lemma used in the proof of the next theorem is actually a special case of it.

LEMMA 4.1. *Let X be a closed subset of s , $[a, b] \subset (0, 1)$ and D be an open subset of $s \times (0, 1)$ containing $X \times [a, b]$. Then there exists an isotopy H of $s \times (0, 1) \times I$ onto $s \times (0, 1)$ such that $H_t(s \times (0, 1)) = s \times (0, 1)$, $H_0 = \text{id}$, for $x \in X$, $H_1(x, a) = (x, b)$ and $H_t| (s \times (0, 1) \setminus D) = \text{id}$. Moreover, if \mathcal{U} is some open cover of D such that for each $x \in X$ there exists $U \in \mathcal{U}$ containing $\{x\} \times [a, b]$, then H may be chosen such that $H|D \times I$ is limited by \mathcal{U} .*

PROOF. By Lemma 2.2 there exist continuous functions f and g from s into $(0, 1)$ such that $W \subset \bigcup_{x \in X} V_x$ and for each $x \in X$, $f(x) < a$ and $b < g(x)$, and $\{x\} \times [f(x), g(x)] \subset U$ for some $U \in \mathcal{U}$. Hence, for each $x \in X$, there exists an open set V_x of x in s such that $V_x \times [f(x), g(x)] \subset U$ for some $U \in \mathcal{U}$. Let W be an open set containing X such that for each $x \in \text{Cl } W$, $f(x) < a$ and $b < g(x)$. Now for each $x \in \text{Cl } W$ and $t \in [0, 1]$ define $\eta_{x,t}: (0, 1) \rightarrow (0, 1)$ by

$$\begin{aligned} \eta_{x,t}(r) &= \frac{a + t(b - a) - f(x)}{a - f(x)} r - \frac{t(b - a)f(x)}{a - f(x)}, & f(x) \leq r \leq a, \\ &= \frac{g(x) - a - t(b - a)}{g(x) - a} r + \frac{t(b - a)g(x)}{g(x) - a}, & a \leq r \leq g(x), \\ &= r, & \text{otherwise.} \end{aligned}$$

Let $\phi: s \rightarrow [0, 1]$ be such that $\phi|X = 1$, $\phi|s \setminus W = 0$. For each $t \in [0, 1]$ define a homeomorphism H_t of $s \times (0, 1)$ onto itself by

$$\begin{aligned} H_t(x, r) &= (x, \eta_{x,t\phi(x)}(r)), & x \in \text{Cl } W, \\ &= (x, r), & \text{otherwise.} \end{aligned}$$

The family of maps $\{H_t\}$ yields the desired isotopy. We verify that H is limited by \mathfrak{u} . Observe that for each $x \notin W$, $H_t(x, r) = (x, r)$ for each $t \in [0, 1]$ and $r \in (0, 1)$. For $x \in \text{Cl}W$ and $r \notin [f(x), g(x)]$, $H_t(x, r) = (x, r)$; for $r \in [f(x), g(x)]$, $H_t(x, r) \in [f(x), g(x)]$. Therefore $\{H_t\}$ is limited by \mathfrak{u} .

In the following theorem, we use $\text{st}^{(n)}\mathfrak{u}$ to denote the n th star of the cover \mathfrak{u} where the star of

$$\mathfrak{u} = \left\{ V : \text{for some } U \in \mathfrak{u}, V = \bigcup_{W \in \mathfrak{u} \text{ and } W \cap U \neq \emptyset} W \right\}.$$

THEOREM 4.2. *If f and g are embeddings of X into M such that $f(X)$ and $g(X)$ are Z -sets, then there exists an ambient isotopy G of M onto itself such that $G_0 = \text{id}$ and $G_1 \circ f = g$ if and only if f and g are homotopic. Moreover, if H is a homotopy between f and g and \mathfrak{u} is some cover of M for which H is limited by \mathfrak{u} , then G may be chosen such that G is limited by $\text{st}^{(4)}\mathfrak{u}$.*

PROOF. Necessity is obvious since the isotopy implies the homotopy. For sufficiency we shall first assume that $f(X)$ and $g(X)$ are disjoint. Let H be a homotopy between f and g . Then $H|X \times \{0\} \cup X \times \{1\}$ is an embedding of $X \times \{0\} \cup X \times \{1\}$ onto a Z -set in M so, by Theorem 3.1, H may be replaced by an embedding of $X \times I$ onto a Z -set of M which extends $H|X \times \{0\} \cup X \times \{1\}$ and which is \mathfrak{u} -close to H . Thus we may assume that H is an embedding of $X \times I$ onto a Z -set and is limited by $\text{st } \mathfrak{u}$. Let σ be an open embedding of M in $s \times (0, 1)$ such that $H(X \times I)$ is a Z -set in $s \times (0, 1)$. Let ν be an embedding of X into s such that $\nu(X) \times [a, b]$ is a Z -set in $s \times (0, 1)$. The mapping ρ of $\sigma \circ H(X \times I)$ to $\nu(X) \times [a, b]$ defined by $\rho(\sigma \circ H(x, t)) = (\nu(x), (b-a)t+a)$ is a homeomorphism of $\sigma \circ H(X \times I)$ onto $\nu(X) \times [a, b]$. Since each of these is a Z -set in $s \times (0, 1)$ (which is homeomorphic to s), by Corollary 10.3 of [3], ρ may be extended to a homeomorphism of $s \times (0, 1)$ onto itself. We shall denote the extension by ρ also. By Lemma 4.1 there exists an isotopy $\{F_t\}$ of $s \times (0, 1)$ onto itself such that $F_t|s \times (0, 1) \setminus \rho \circ \sigma(M) = \text{id}$, $F_0 = \text{id}$, $F_1(\nu(x), a) = b$, and $F| \rho \circ \sigma(M) \times I$ is limited by the open cover \mathfrak{v} of $\rho \circ \sigma(M)$ induced by $\rho \circ \sigma(M)$ induced by $\rho \circ \sigma$ and $\text{st}(\mathfrak{u})$. Then the isotopy G of M onto itself defined by $G_t = \sigma^{-1} \circ \rho^{-1} \circ F_t \circ \rho \circ \sigma$ satisfies

- (i) $G_0 = \text{id}$,
- (ii) $G_1 \circ f = g$,
- (iii) G is limited by $\text{st } \mathfrak{u}$.

Now consider the general case where $f(X)$ and $g(X)$ are not assumed to be disjoint. By Lemma 2.4 we may replace the homotopy

H by an isotopy F which is \mathfrak{u} -close to H such that $F_0 = H_0$, $F_1 = H_1$, and $F_{1/2}(X) \cap (f(X) \cup g(X))$ are disjoint. Then F is limited by $\text{st } \mathfrak{u}$. By what we have just shown, there exists isotopies $\{G_i^{(1)}\}$ and $\{G_i^{(2)}\}$ of M onto itself, each of which is limited by $\text{st}^2(\mathfrak{u})$ and satisfying $G_0^{(1)} = \text{id}$, $G_1^{(1)} \circ f = F_{1/2}$, $G_0^{(2)} = \text{id}$, $G_1^{(2)} \circ F_{1/2} = g$. In the obvious manner, the isotopy G is obtained satisfying the conclusion of the theorem.

COROLLARY 4.3. *A homeomorphism f of a Z -set K_1 in M onto a set K_2 of M can be extended to a homeomorphism of M onto itself if and only if K_2 is a Z -set and there exists a homeomorphism h of M onto itself such that $h \circ f$ is homotopic to the identity on M .*

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