CONVEX MEROMORPHIC MAPPINGS AND RELATED FUNCTIONS

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1. **Introduction.** Let $U(p)$ denote the class of univalent meromorphic functions $f(z)$ in the unit disk E with a simple pole at $z=p>0$ and with the normalization $f(0)=0$ and $f'(0)=1$. Let $K(p)$ denote the functions which belong to $U(p)$ and map $|z|<r>\rho$ (for some $p<\rho<1$) onto the complement of a convex set. If $f(z)\in K(p)$, then there is a ρ, $p<\rho<1$, such that for each z, $\rho<|z|<1$

$$\mathrm{Re}\{1+zf''(z)/f'(z)\}\leqq 0.$$

If $f\in K(p)$, then for each z in E,

$$\mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}+\frac{2p}{z-p}-\frac{2pz}{1-pz}\right\}\leqq 0. \tag{1}$$

Let $\Sigma(p)$ denote the class of mappings $f(z)$ which satisfy (1) and the conditions $f(0)=0$ and $f'(0)=1$. The class $K(p)$ is contained in $\Sigma(p)$. Royster [9] has shown that for $0<p<2-\sqrt{3}$, if $f(z)\in\Sigma(p)$ and is meromorphic, then $f(z)\in K(p)$. The class $U(p)$ and related classes have been studied in [1], [2], [4], and [5].

In this paper, we shall be interested in examining functions in $K(p)$ and $\Sigma(p)$ with respect to their properties near the origin. In particular, we shall examine the radius of convexity of $K(p)$ and $\Sigma(p)$. We shall also define and study close-to-convex functions and functions which are starlike with respect to some point in the complement of $f(0)$.

2. **Convex functions.** Let $\mathcal{P}$ denote the class of functions $P(z)$ which are regular in E and satisfy $P(0)=1$ and $\mathrm{Re}\{P(z)\}\geqq 0$ for all $z\in E$. For each function $f(z)\in\Sigma(p)$, there is a function $P(z)\in\mathcal{P}$ such that

$$1+z\frac{f''(z)}{f'(z)}+\frac{2p}{z-p}-\frac{2pz}{1-pz}=-P(z). \tag{2}$$

Taking real parts, we have

$$\mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\}=2p\,\mathrm{Re}\left\{\frac{z}{1-pz}-\frac{1}{z-p}\right\}-\mathrm{Re}\{P(z)\}. \tag{3}$$

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To find the radius of convexity for $\Sigma(p)$, we need the following lemma:

LEMMA 1. *Let*

$$Q(r,\theta) = \operatorname{Re}\left\{\frac{z}{1-pz} - \frac{1}{z-p}\right\}$$

where $z=re^{i\theta}$ *and* $0<p<1$. *Then for* $r<p$, *we have*

$$\min_{0\leqq\theta<2\pi} Q(r,\theta) = Q(r,\pi) = \frac{1-r^2}{(p+r)(1+pr)}. \tag{4}$$

PROOF. Computing $Q(r,\theta)$ yields

$$Q(r,\theta) = \frac{r\cos\theta - pr^2}{1-2pr\cos\theta + p^2r^2} + \frac{p - r\cos\theta}{r^2 - 2pr\cos\theta + p^2}.$$

For a fixed r, we have

$$\frac{dQ}{d\theta} = \frac{-r\,q(r,\theta)\sin\theta}{(1-2pr\cos\theta+p^2r^2)^2(r^2-2pr\cos\theta+p^2)^2}$$

where

$$q(r,\theta) = (1-p^2r^2)(r^2-2pr\cos\theta+p^2)^2 + (p^2-r^2)(1-2pr\cos\theta+p^2r^2)^2.$$

If $r<p$, we see that $q(r,\theta)>0$ for all θ, $0\leqq\theta<2\pi$. Thus for $r<p$, $Q(r,\theta)$ will have its minimum at $\theta=\pi$. This proves the lemma.

THEOREM 1. *If* $f(z)\in\Sigma(p)$, *then* $f(z)$ *maps* $|z|<\rho_0(p)$ *onto a convex set, where*

$$\rho_0(p) = [1+4p+p^2-(p+1)(p^2+6p+1)^{1/2}]/2p.$$

PROOF. For $P(z)\in\mathcal{P}$, it is well known that

$$\max_{P\in\mathcal{P}}\max_{|z|=r}\operatorname{Re}\{P(z)\} = \frac{1+r}{1-r}. \tag{5}$$

If $f(z)$ satisfies the hypotheses of the theorem, then (3), (4), and (5) give

$$\begin{aligned}\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} &\geqq \frac{2p(1-r^2)}{(p+r)(1+pr)} - \frac{1+r}{1-r} \\ &= \frac{p-(1+3p+p^2)(r+r^2)+pr^3}{(p+r)(1+pr)(1-r)}\end{aligned} \tag{6}$$

for $|z|<p$. This proves the theorem.

A brief calculation shows that a function $f(z)\in\Sigma(p)$ is meromorphic in E if and only if the function $P(z)$ given in (2) has the properties

(i) $P(z)\in\mathcal{P}$ and

(ii) $P(p)=(1+p^2)(1-p^2)^{-1}$.

Thus for a function $f(z)\in K(p)$, equation (2) holds where the function $P(z)$ satisfies (i) and (ii).

To find the radius of convexity for $K(p)$, first we must prove the following lemma.

LEMMA 2. *Let* $P(z)\in\mathcal{P}$ *and* $P(p)=(1+p^2)(1-p^2)^{-1}$, $0<p<1$. *Then for* $r<p/2$,

$$\max_{|z|=r} \operatorname{Re}\{P(z)\} \leqq \frac{(1+p^2)(1-r^2)}{(1-rp)^2+(r-p)^2}.$$

PROOF. Let $w(z)=(P(z)-1)(P(z)+1)^{-1}$. Then $w(z)$ is regular in E and satisfies $w(0)=0$, $w(p)=p^2$, and $|w(z)|<1$ for $z\in E$. Define the following functions:

$$g(z)=\frac{w(z)-p^2}{1-p^2w(z)}\cdot\frac{1-pz}{z-p}, \qquad h(z)=-\frac{g(z)-p}{1-pg(z)}.$$

Then $h(z)$ is regular in E, $h(0)=0$, and $|h(z)|<1$. Thus $|h(z)|\leqq z$. Let $\zeta=(z-p)(1-pz)^{-1}$, then we may write

$$P(z)=\frac{1+p^2}{1-p^2}\cdot\frac{1+g(z)\zeta}{1-g(z)\zeta}.$$

Now, the problem max $\operatorname{Re}\{P(z)\}$ is equivalent to

$$\max_{|z|=r} \operatorname{Re}\left\{\frac{1+g(z)\zeta}{1-g(z)\zeta}\right\}=\max_{|z|=r}\frac{1-|g(z)\zeta|^2}{|1-g(z)\zeta|^2}.$$

Let z satisfy $|z|=r$; then $\zeta=(z-p)(1-pz)^{-1}$ lies on the circle C_1: $|\zeta+a|=R$ where $a=p(1-r^2)(1-p^2r^2)^{-1}$ and $R=r(1-p^2)(1-r^2p^2)^{-1}$. Since $|h(z)|\leqq|z|$, the point $g(z)$ lies in the closed disk C_2: $|g(z)-a|\leqq R$. The quantity $|g(z)\zeta|$ is minimum when $g(z)=a-R$ and $\zeta=-a+R$. We will show that max $\operatorname{Re}\{P(z)\}$ occurs at the same point as $\min|g(z)\zeta|$ provided $R<a/2$. Let $\zeta_0=-a+Re^{i\theta}$ lie on C_1. Then $g(z)\zeta_0$ lies in the circle C_3: $|g(z)\zeta_0-\zeta_0 a|\leqq|\zeta_0|R$. The distance from 1 to the circle is $d(\theta)=|1-\zeta_0 a|-|\zeta_0|R$. We observe that $\min_{0\leqq\theta<2\pi} d(\theta)=\min_{|z|=r}|1-g(z)\zeta|$. The function $d(\theta)$ is a minimum at $\theta=0$ provided $|a^2R^2-a^2(1+a^2)|>2|(1+a^2)aR^3-aR|$ which holds for $R<a/2$. Thus for $r<p/2$ we have shown

$$\frac{1-|g(z)\zeta|^2}{|1-g(z)\zeta|^2} \leqq \frac{1-(a-R)^2}{1+(a-R)^2} = \frac{(1-p^2)(1-r^2)}{(1-pr)^2+(p-r)^2}$$

which proves the lemma. Equality occurs when $h(z)=z$ at $z=r$.

Applying Lemmas 1 and 2 to equation (3), we have the following:

THEOREM 2. *If $f(z)\in K(p)$, then*

$$\operatorname{Re}\{1+z(f''(z)/f'(z))\} > 0$$

for $|z| < \min[p/2, \rho_1(p)]$ where $\rho_1(p)$ is the smallest positive root of

$$p^3+p-(1+10p^2+p^4)r+(p^3+p)r^2 = 0.$$

For $.1 < p < 1$, we have $\rho_1(p) < p/2$.

The number $\rho_0(p)$ for $\Sigma(p)$ in Theorem 1 is best possible, whereas the number $\rho_1(p)$ in Theorem 2 is not. As p tends to 1, both $\rho_0(p)$ and $\rho_1(p)$ tend to $3-2\sqrt{2}$.

3. **Close-to-convex functions.** Let us define a class of close-to-convex functions. A function $f(z)(f(0)=0, f'(0)=1)$ belongs to the class $C\Sigma(p)$ (close-to-convex functions) if $f(z)$ is regular at each point of E except at p and if there is a function $e^{i\Phi}g(z)\in\Sigma(p)$, $|\Phi|\leqq\pi/2$, such that

$$\operatorname{Re}\{f'(z)/g'(z)\} > 0$$

for all $z\in E$. Let $C\Sigma_0(p)$ denote the subclass of $C\Sigma(p)$ for which $\Phi=0$. A function in $C\Sigma(p)$ may have a logarithmic singularity at $z=p$ even though the corresponding function $g(z)$ is meromorphic in E.

Let $f(z)\in C\Sigma(p)$ and $e^{i\Phi}g(z)\in\Sigma(p)$ be functions such that

$$\operatorname{Re}\{f'(z)/g'(z)\} \geqq 0 \tag{7}$$

for all $z\in E$. Define the functions $P_1(z)$ and $Q_1(z)$ such that $i\exp[iP_1(z)]$ and $i\exp[iQ_1(z)]$ are the unit tangent vectors to the images of $|z|=r$ under $f(z)$ and $g(z)$, respectively. Since (7) holds, we have (provided the proper choice of arguments has been made)

$$|P_1(z)-Q_1(z)| < \pi/2.$$

Define $Q(r,\theta)=Q_1(z)+\arg[(z-p)^2(1-pz)^2z^{-2}]$ and $P(r,\theta)=P_1(z)+\arg[(z-p)^2(1-pz)^2z^{-2}]$ where $z=re^{i\theta}$. Since $e^{i\Phi}g(z)\in\Sigma(p)$, we have $\partial Q/\partial\theta<0$. An argument similar to one used by Kaplan [3] yields

$$P(r,\theta_1)-P(r,\theta_2) > -\pi \quad \text{for } \theta_1<\theta_2,$$

which is equivalent to

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta$$

(8)

$$< \pi - \arg\left[(re^{i\theta_2} - p)^2(1 - pre^{i\theta_2})^2 r^{-2} e^{-2i\theta_2}\right]$$
$$+ \arg\left[(re^{i\theta_1} - p)^2(1 - pre^{i\theta_1})^2 r^{-2} e^{-2i\theta_1}\right].$$

Thus (7) implies (8).

To prove the converse, we define $P(r, \theta)$ as defined above and assume that (8) holds. Since $f(z)$ may have a logarithmic singularity, we cannot use Libera and Robertson's proof, although the proof given here is similar to both their proof and Kaplan's. From the definition of $P(r, \theta)$, we see $P(r, \theta+2\pi)-P(r, \theta)=-2\pi$. We now use

LEMMA 3 [6]. *Let $t(\theta)$ be a real function of θ for $-\infty<\theta<\infty$ such that*

$$t(\theta + 2\pi) - t(\theta) = -2\pi,$$
$$t(\theta_1) - t(\theta_2) > -\pi \quad \text{for } \theta_1 > \theta_2.$$

Then there exists a real valued function $s(\theta)$ which is nonincreasing and satisfies the conditions

$$s(\theta + 2\pi) - s(\theta) = -2\pi,$$
$$|s(\theta) - t(\theta)| \leqq \pi/2.$$

For a fixed ρ, let $t(\theta)=P(\rho, \theta)$ and denote a corresponding $s(\theta)$ which is given in Lemma 3 by $s(\rho, \theta)$. For $r<\rho$, define $q_\rho(r, \theta)$ by

$$q_\rho(r, \theta) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{s(\rho, \alpha) + \alpha}{\rho^2 - 2\rho r \cos(\theta - \alpha) + r^2} d\alpha.$$

The function $q_\rho(r, \theta)$ is harmonic for $r<\rho$. Now define

$$Q_\rho(r, \theta) = q_\rho(r, \theta) - \theta.$$

Since $s(\rho, \alpha+\theta_2)-s(\rho, \alpha+\theta_1)$ is nonpositive for $\theta_1<\theta_2$ and has period 2π, we may write

$$Q_\rho(r, \theta_2) - Q\rho(r, \theta_1) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{s(\rho, \alpha + \theta_2) - s(\rho, \alpha + \theta_1)}{\rho^2 - 2\rho r \cos \alpha + r^2} d\alpha < 0.$$

Let $h_\rho(z)$ be the analytic completion of $q_\rho(r, \theta)$ such that $\operatorname{Im} h_\rho(z) =q_\rho(r, \theta)$ and $\operatorname{Re} h_\rho(0)=0$. Define $g_\rho(z)$ to be

$$g_\rho(z) = p^2 \int_0^z \frac{e^{h_\rho(z)}}{(z-p)^2(1-pz)^2}\, dz.$$

If $|z| < \rho$, then $g_\rho(z)$ satisfies

$$|g_\rho'(0)| = 1,$$

$$\operatorname{Re}\left\{1 + z\frac{g''(z)}{g'(z)} + \frac{2p}{z-p} - \frac{2pz}{1-pz}\right\} = \operatorname{Re}\{zh_\rho'(z) - 1\}$$

$$= \frac{\partial Q_\rho(r,\theta)}{\partial\theta} < 0,$$

$$\operatorname{Re}\{f'(z)/g_\rho'(z)\} > 0.$$

The function $F_\rho(z) = e^{h_\rho(z)}/z$ is a univalent starlike function in $0<|z|<\rho$. Following the argument of Kaplan [3] we can choose a sequence $\rho_n \to 1$ such that F_{ρ_n} converges uniformly to $F(z)$ in every closed domain of E. The function $F(z)$ is univalent and starlike in $0<|z|<1$, and the function

$$g'(z) = p^2 \frac{zF(z)}{(z-p)^2(1-pz)^2}$$

satisfies (1) and

$$\operatorname{Re}\{f'(z)/g'(z)\} > 0$$

for all $|z|<1$. Thus (8) is an equivalent condition for a function to belong to $C\Sigma(\rho)$.

REMARK. Suppose $p=0$; then condition (1) for $g(z)$ is

$$\operatorname{Re}\{1 + zg''(z)/g'(z)\} < 0.$$

Thus $F(z) = zg'(z)$ is starlike with respect to the origin in $0<|z|<1$. Equations (7) and (8) become

$$\operatorname{Re}\{zf'(z)/f(z)\} > 0$$

and

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta < \pi(\theta_1 < \theta_2)$$

which are the conditions Libera and Robertson [6] derived for close-to-convex functions in $0<|z|<1$.

THEOREM 3. *Let $f(z) \in C\Sigma_0(p)$; then $f(z)$ is convex and univalent in $|z| < \rho_2$, where ρ_2 is the smallest positive root of*

$$p(1 + r^4) - (1 + 4p^2)(r + r^3) - 2(3 + 2p + 3p^2)r^2 = 0.$$

PROOF. For a function $f(z) \in C\Sigma_0(p)$ there exist functions $P(z) \in \mathcal{P}$ and $g(z) \in \Sigma(p)$ such that

$$f'(z) = g'(z)P(z)$$

for $|z| < 1$. Thus we have

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} = \operatorname{Re}\left\{1 + z\frac{g''(z)}{g'(z)}\right\} + \operatorname{Re}\left\{z\frac{P'(z)}{P(z)}\right\}.$$

Using (6) and a result of MacGregor [7], we have

$$\begin{aligned}\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} &\geqq \frac{p - (1 + 3p + p^2)(r + r^2) + pr^3}{(p + r)(1 + pr)(1 - r)} - \frac{2r}{1 - r^2} \\ &= \frac{p(1 + r^4) - (1 + 4p + p^2)(r + r^3) - 2(3 + 2p + 3p^2)r^2}{(1 - r^2)(p + r)(1 + pr)}\end{aligned}$$

for $z = re^{i\theta}$ and $r < p$. This completes the proof.

A corresponding theorem may be proved for the case where the function $g(z)$ in (7) is meromorphic. If the function $g(z) \in K(p)$, then (8) may be replaced with

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta < \pi (\theta_1 < \theta_2)$$

for $r > \rho$, where ρ is such that if $|z| > \rho$, then

$$\operatorname{Re}\{1 + zg''(z)/g'(z)\} \leqq 0.$$

4. **Starlike functions.** Since functions in $U(p)$ map the disk onto the exterior of a bounded domain, the concept of starlike with respect to the origin or the point at infinity cannot be used. However, we may use Robertson's concept of star center points [8]. Let $U^*(p, w_0)$ denote the set of functions defined by: $f(z) \in U^*(p, w_0)$ if and only if $f(z) \in U(p)$, $f(z) \neq w_0$ for all $z \in E$, and there exists a ρ, $p < \rho < 1$, such that

$$\operatorname{Re}\{zf'(z)/(f(z) - w_0)\} \leqq 0 \tag{9}$$

for $\rho < |z| < 1$. Functions in $U^*(p, w_0)$ map E onto the exterior of sets which are starlike with respect to w_0.

THEOREM 4. *Let* $f(z)\in U^*(p, w_0)$ *and* $0<p<2-\sqrt{3}$. *Then*

$$\operatorname{Re}\{zf'(z)/(f(z)-w_0)\} \leqq 0$$

for all $|z|>\rho=[1-6p^2+p^4-(1-p^2)(p^2-14p^2+1)^{1/2}]\cdot[2p+2p^3]^{-1}$.

PROOF. If (9) holds, then there exists a function $P(z)\in\mathcal{P}$ such that

$$z\frac{f'(z)}{f(z)-w_0}+\frac{p}{z-p}-\frac{pz}{1-pz}=-P(z).$$

Taking real parts, we get

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)-w_0}\right\}=p\operatorname{Re}\left\{\frac{z}{1-pz}-\frac{1}{z-p}\right\}-\operatorname{Re}\{P(z)\}.$$

We now need the following lemma:

LEMMA 4 [9]. *Let* $Q(r,\theta)$ *be as defined in Lemma* 1, *and*

$$R(p)=\frac{1-6p^2+p^4-(1-p^2)(p^4-14p^2+1)^{1/2}}{2p+2p^3}.$$

If $0<p<2-\sqrt{3}$ *and* $R(p)<r<1$, *then*

$$\max_{0\leqq\theta<2\pi}\operatorname{Re}\{Q(r,\theta)\}=\operatorname{Re}Q(r,\pi)=\frac{(1-r^2)}{(r+p)(1+rp)}.$$

Using Lemma 4, we have

$$\begin{aligned}\operatorname{Re}\left\{z\frac{f'(z)}{f(z)-w_0}\right\}&\leqq\frac{p(1-r^2)}{(r+p)(1+pr)}-\frac{1-r}{1+r}\\&=-\frac{(1-r)(1-p)^{1/2}r}{(1+r)(r+p)(1+pr)}<0\end{aligned}$$

for all $|z|=r>R(p)$.

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