CHARACTERS ON SINGLY GENERATED C^*-ALGEBRAS[1]

JOHN BUNCE

ABSTRACT. In this note we consider the question of what elements δ in the spectrum of a bounded operator A on Hilbert space have the property that there is a multiplicative linear functional ϕ on the C^*-algebra generated by A and I whose value at A is δ. If A is hyponormal then there is a character ϕ on the C^*-algebra generated by A and I such that $\phi(A)=\delta$ if and only if δ is in the approximate point spectrum of A. We use this to prove a structure theorem for the C^*-algebra generated by a hyponormal operator. We conclude by proving that any pure state on a Type I C^*-algebra is multiplicative on some maximal abelian C^*-subalgebra.

Let A be an operator in $B(H)$, the set of bounded operators on a Hilbert space H. Let $C^*(A)$ be the C^*-subalgebra of $B(H)$ generated by A and the identity. By a character on a C^*-algebra we mean a multiplicative linear functional. We will investigate the existence of characters on $C^*(A)$. Let $\mathrm{sp}(A)$ denote the spectrum of A. The author began this research after learning that William Arveson had proved the following: If $\delta \in \mathrm{sp}(A)$ is such that $\|A\| = |\delta|$, then there is a character ϕ on $C^*(A)$ such that $\phi(A)=\delta$. We independently prove this result at the end of this note. We denote by $a(A)$ the approximate point spectrum of A; i.e., $a(A)$ is the set of scalars δ for which there is a sequence (x_n) of unit vectors in H such that $(A-\delta I)x_n$ converges to zero in norm. Let $p(A)$ denote the set of eigenvalues of A. See [6, §31] for facts about spectra.

PROPOSITION 1. *Let $A \in B(H)$. Then*

$$a(A) = \{\delta \in \mathrm{sp}(A) : C^*(A)(A - \delta I) \neq C^*(A)\}.$$

PROOF. First let $\delta \in a(A)$, then $A-\delta I$ is not bounded below. If I were in $C^*(A)(A-\delta I)$, then we would have $I = D(A-\delta I)$ for some

Received by the editors, July 22, 1969.

AMS Subject Classifications. Primary 4710, 4665.

Key Words and Phrases. Character, approximate point spectrum, irreducible representation, universal representation, pure state, atomic representation, Radon-Nikodým theorem, hyponormal operator, Cesàro operator, irreducible operator, weighted shift, maximal abelian subalgebras.

[1] This is part of the author's doctoral dissertation at Tulane University under the direction of Dr. David M. Topping. The research was supported by a NSF research assistantship.

$D\in C^*(A)$, and for $x\in H$, $\|x\|\leqq\|D\|\,\|(A-\delta I)x\|$; hence $A-\delta I$ would be bounded below. Thus $I\notin C^*(A)(A-\delta I)$. Conversely, suppose δ is such that $C^*(A)(A-\delta I)\neq C^*(A)$. Then there is a pure state ϕ on $C^*(A)$ such that $C^*(A)(A-\delta I)$ is contained in $K(\phi)$, where $K(\phi)=\{B\in C^*(A):\phi(B^*B)=0\}$, [**3**, 2.9.5]. If we had $\delta\notin a(A)$, then $A-\delta I$ would be bounded below and there would exist $m>0$ such that $m^2I\leqq(A-\delta I)^*(A-\delta I)$. But $\phi((A-\delta I)^*(A-\delta I))=0$, so $m^2\phi(I)=0$, but this is a contradiction. Hence $\delta\in a(A)$.

The next two corollaries say roughly that to compute the approximate point spectrum of an element we can use any C^*-algebra containing the element.

COROLLARY 2. *Let $\delta\in\mathrm{sp}(A)$. Then if $I\in B(H)(A-\delta I)$, we also have $I\in C^*(A)(A-\delta I)$.*

PROOF. If $I=D(A-\delta I)$ for some $D\in B(H)$, then as above $(A-\delta I)$ is bounded below so $\delta\notin a(A)$, and by the proposition $C^*(A)(A-\delta I)$ $=C^*(A)$.

COROLLARY 3. *Let $A\in B(H)$. Then*

$$a(A)=\{\delta\in\mathrm{sp}(A):B(H)(A-\delta I)\neq B(H)\}.$$

For facts about irreducible representations see [**3**, §2].

COROLLARY 4. *If $\delta\in a(A)$, then there exists a pure state ϕ on $C^*(A)$ such that $\pi_\phi(A)\xi_\phi=\delta\xi_\phi$, where π_ϕ is the irreducible representation induced by ϕ, with canonical cyclic vector ξ_ϕ.*

PROOF. If $\delta\in a(A)$, then $C^*(A)(A-\delta I)\neq C^*(A)$; so there is a pure state ϕ on $C^*(A)$ such that $C^*(A)(A-\delta I)\subset K(\phi)$. Then $A+K(\phi)$ $=\delta I+K(\phi)$, so $\pi_\phi(A)\xi_\phi=\delta\xi_\phi$.

COROLLARY 5. *If $\delta\notin a(A)$, then $\delta\notin a(\pi(A))$ for any representation π of $C^*(A)$ on a Hilbert space. (We assume $\pi(I)=I$.)*

PROOF. If $\delta\notin a(A)$, then $I=D(A-\delta I)$ for some $D\in C^*(A)$, so $\pi(I)=I=\pi(D)(\pi(A)-\delta I)$. Hence $\delta\notin a(\pi(A))$.

For facts about the universal representation see [**3**, 2.7.8].

COROLLARY 6. *Let $\pi_0:C^*(A)\rightarrow B(H_0)$ be the universal representation of $C^*(A)$. Then $a(A)=p(\pi_0(A))=a(\pi_0(A))$ and $a(A^*)=p(\pi_0(A^*))$ $=a(\pi_0(A^*))$.*

PROOF. First note that by Proposition 1, $a(A)$ depends only on $C^*(A)$ and hence $a(A)=a(\pi_0(A))$. But by Corollary 4, $a(A)\subset p(\pi_0(A))$

$\subset a(\pi_0(A))=a(A)$. So $a(A)=p(\pi_0(A))=a(\pi_0(A))$. Since $C^*(A)=C^*(A^*)$, we also have $a(A^*)=p(\pi_0(A^*))=a(\pi_0(A^*))$.

The conclusion of Corollary 6 holds if we take for π_0 the *atomic* representation of $C^*(A)$; i.e., the direct sum of all irreducible representations [**5**, p. 388].

PROPOSITION 7. *Let $A\in B(H)$ and $\delta\in \mathrm{sp}(A)$. Then there is a pure state ϕ on $C^*(A)$ such that $\phi(A)=\delta$.*

PROOF. First suppose $\delta\in a(A)$ and let ϕ be a pure state on $C^*(A)$ such that $C^*(A)(A-\delta I)\subset K(\phi)$. Then $\phi(A)=\delta$. If $\delta\notin a(A)$, then $A-\delta I$ does not have dense range and $A^*-\bar{\delta}I$ is not one-to-one. So $\bar{\delta}\in a(A^*)$ and as above there is a pure state ϕ on $C^*(A)$ such that $\phi(A^*)=\bar{\delta}$, or $\phi(A)=\delta$.

PROPOSITION 8. *If $A\in B(H)$ and ϕ is a character on $C^*(A)$, then $\phi(A)\in a(A)$.*

PROOF. Clearly $A-\phi(A)I\in\phi^{-1}(0)$, which is a proper two-sided ideal of $C^*(A)$. So $\phi(A)\in\mathrm{sp}(A)$. Also $C^*(A)(A-\phi(A)I)\subset\phi^{-1}(0)\neq C^*(A)$, so by Proposition 1, $\phi(A)\in a(A)$.

PROPOSITION 9. *Let $A\in B(H)$ and let $\delta\in a(A)$ be such that $(A-\delta I)\cdot(A^*-\bar{\delta}I)\leqq N^2(A^*-\bar{\delta}I)(A-\delta I)$ for some $N>0$. Then there exists a character ϕ on $C^*(A)$ such that $\phi(A)=\delta$.*

PROOF. By the Radon-Nikodým theorem for operators [**4**], there exists an operator $D\in B(H)$ such that $A-\delta I=(A^*-\bar{\delta}I)D$, or $A^*-\bar{\delta}I=D^*(A-\delta I)$. Hence $B(H)(A^*-\bar{\delta}I)\subset B(H)(A-\delta I)$. Since $\delta\in a(A)$, $B(H)(A-\delta I)$ is a proper left ideal by Corollary 3. Hence there is a pure state ϕ on $B(H)$ such that $B(H)(A^*-\bar{\delta}I)\subset B(H)(A-\delta I)\subset K(\phi)$. Then for all $B\in B(H)$ we have $\phi(BA^*)=\phi(B)\bar{\delta}=\phi(B)\phi(A^*)$ and $\phi(BA)=\phi(B)\phi(A)$. Hence if $M(A, A^*)$ is any monomial in A and A^*, we have that $\phi(M(A,A^*))=M(\phi(A),\phi(A^*))$. Hence ϕ is a character on $C^*(A)$ and $\phi(A)=\delta$.

An operator A on H is called *hyponormal* if $AA^*\leqq A^*A$. If A is hyponormal, then $A-\delta I$ is also hyponormal for all scalars δ. Then as an immediate corollary of Proposition 9 we have

COROLLARY 10. *Let $A\in B(H)$ be hyponormal. Then for all $\delta\in a(A)$, there exists a character ϕ on $C^*(A)$ such that $\phi(A)=\delta$.*

COROLLARY 11. *Let $A\in B(H)$ be hyponormal. Let S be the set of all nonzero characters on $C^*(A)$. Let $I=\bigcap\{\phi^{-1}(0):\phi\in S\}$. Then $C^*(A)/I$ is isomorphic to the C^*-algebra of continuous complex-valued functions on $a(A)$.*

Proof. By Corollary 10, S is nonempty. So I is a closed two-sided ideal, and $C^*(A)/I$ is a C^*-algebra with identity which is clearly commutative. So $C^*(A)/I$ is isomorphic to $C(X)$ for some compact Hausdorff space X. But $C^*(A)/I$ is singly generated by the normal element $A+I$, hence $X=\mathrm{sp}(A+I)=\{\bar{\phi}(A+I):\bar{\phi}$ is a character of $C^*(A)/I\}$. Let $\pi: C^*(A)\to C^*(A)/I$ be the natural projection. Then clearly the map $\bar{\phi}\to\bar{\phi}\circ\pi$ is a bijection between the set of characters on $C^*(A)/I$ and the set S of characters on $C^*(A)$. So $X=\{\phi(A): \phi\in S\}$. Then by Corollary 10, $a(A)\subset X$ and by Proposition 8, $X\subset a(A)$. So $X=a(A)$.

In certain cases, Corollary 10 may be used to compute the approximate point spectrum of an operator. For example, consider the discrete Cesàro operator on l^2 [2], C_0, given by

$$(C_0x)(n) = 1/(n+1)(x_0+x_1+\cdots+x_n), \qquad \text{for } n=0,1,2,\cdots,$$

where $x=(x_0, x_1, x_2, \cdots)$ is in l^2. Brown, Halmos, and Shields [2] proved that $\|C_0\|=2$, $\mathrm{sp}(C_0)=\{\delta: |1-\delta|\leqq 1\}$, the point spectrum of C_0 is empty, and C_0 is hyponormal. Computing the approximate point spectrum of C_0 directly seems somewhat difficult, but using Corollary 10 we can show that $a(C_0)=\{\delta: |1-\delta|=1\}$. If $|1-\delta|=1$, then δ is in the boundary of $\mathrm{sp}(C_0)$, so $\delta\in a(C_0)$ [7, Problem 63]. Conversely let $\delta\in a(C_0)$, then there is a character ϕ on $C^*(C_0)$ such that $\phi(C_0)=\delta$. Now it is known and easily shown that $C_0+C_0^*-C_0C_0^*=D_0$ is a diagonal operator with diagonal $1, 1/2, 1/3, \cdots$ and is hence compact. Now if an operator A commutes with C_0 and C_0^*, then A commutes with D_0 and is thus a diagonal operator. Also, it is easily seen that the only diagonal operator commuting with C_0 is a scalar, so A is a scalar, and C_0 is hence irreducible. Thus by [3, 4.1.10] $C^*(C_0)$ contains the compact operators on l^2. But then ϕ is a character on the simple C^*-algebra of compact operators on l^2, so ϕ vanishes on the compacts. Thus

$$\phi(C_0)+\overline{\phi(C_0)}-|\phi(C_0)|^2=\phi(D_0)=0.$$

If $\phi(C_0)$ has real part a and imaginary part b, this becomes $2a-a^2-b^2=0$, or $(a-1)^2+b^2=1$. Hence $\phi(C_0)=\delta$ is on the circle of radius 1 and center $(1, 0)$, so $|1-\delta|=1$. So $a(C_0)=\{\delta: |1-\delta|=1\}$.

We remark that the converse of Proposition 9 is not true: Let $(e_n)_0^\infty$ be the canonical orthonormal basis in l^2 and define a weighted shift $D\in B(l^2)$ by $De_n=\delta_n e_{n+1}$, where $\delta_{2n}=(1/2)^{2n}$ for $n\geqq 0$ and $\delta_{2n+1}=(1/3)^{2n+1}$ for $n\geqq 0$. Then $D^*(x_0, x_1, \cdots)=(\delta_0x_1, \delta_1x_2, \delta_2x_3, \cdots)$ and $DD^*(x_0, x_1, \cdots)=(0, \delta_0^2x_1, \delta_1^2x_2, \cdots)$, $D^*D(x_0, x_1, \cdots)=$

$(\delta_0^2 x_0, \delta_1^2 x_1, \cdots)$. Suppose there existed an $N>0$ such that $DD^* \leqq N^2 D^*D$. Then $\delta_j^2 \leqq N^2\delta_{j+1}^2$ for all $j \geqq 0$. But $\delta_{2n}/\delta_{2n+1} = (3/2)^{2n}3$, which is unbounded. So there does not exist $N>0$ such that $DD^* \leqq N^2D^*D$. If $D^*D \leqq N^2DD^*$ then $\|Dx\| \leqq N\|D^*x\|$ for all $x \in l^2$. But $D^*e_0 = 0$ while $De_0 \neq 0$. So there does not exist N such that $D^*D \leqq N^2DD^*$. Now DD^* is clearly compact, so D is compact. By [**7**, Problem 151] D is irreducible, so by [**3**, 4.1.10] $C^*(D)$ is just the scalars plus the compacts. The projection π of $C^*(D)$ onto $C^*(D)$ modulo the compacts is then a character and $\pi(D)=0$. This shows that the converse of Proposition 9 is true for neither D nor D^*.

To prove Arveson's result (Proposition 13 and Corollary 14) we need the following lemma [**9**, p. 8].

LEMMA 12. *If $T \in B(H)$ is such that $\|T\| \leqq 1$ and $x \in H$ is such that $Tx = x$, then $T^*x = x$.*

PROOF. Since $Tx = x$, we have $(x, T^*x) = (Tx, x) = \|x\|^2$. Hence

$$\begin{aligned} \|x - T^*x\|^2 &= \|x\|^2 - (x, T^*x) - (T^*x, x) + \|T^*x\|^2 \\ &= \|x\|^2 - 2\,\mathrm{Re}(x, T^*x) + \|T^*x\|^2 = \|x\|^2 - 2\|x\|^2 + \|T^*x\|^2 \\ &= \|T^*x\|^2 - \|x\|^2 \leqq 0. \end{aligned}$$

So $x = T^*x$.

PROPOSITION 13. *Let $A \in B(H)$ be such that $\|A\| \leqq 1$ and suppose $1 \in \mathrm{sp}(A)$. Then there exists a character ϕ on $C^*(A)$ such that $\phi(A) = 1$.*

PROOF. Since 1 is in the boundary of $\mathrm{sp}(A)$, we have $1 \in a(A)$. Let $\pi_0: C^*(A) \to B(H_0)$ be the universal representation. Then by Corollary 6 we have $a(A) = p(\pi_0(A)) = a(\pi_0(A))$. Hence $1 \in p(\pi_0(A))$ and there is an $x \in H_0$ such that $\pi_0(A)x = x$. Then by the lemma $\pi_0(A^*)x = x$. Then if ω_x is the vector state associated with x, we have for $M(\pi_0(A), \pi_0(A^*))$ any monomial in $\pi_0(A)$ and $\pi_0(A^*)$,

$$\omega_x(M(\pi_0(A), \pi_0(A^*))) = M(1, 1).$$

Hence ω_x is a character on $\pi_0(C^*(A))$, so $\omega_x \circ \pi_0$ is a character on $C^*(A)$ and $\omega_x \circ \pi_0(A) = \|x\|^2 = 1$.

COROLLARY 14. *Let A be a nonzero element of $B(H)$ and suppose there is a $\delta \in \mathrm{sp}(A)$ such that $\|A\| = |\delta|$. Then there is a character ϕ on $C^*(A)$ such that $\phi(A) = \delta$.*

PROOF. Consider $B = \delta^{-1}A$ and use Proposition 13.

We briefly consider an unrelated C^*-algebra problem first raised by Kadison and Singer [**8**]. Let ϕ be a pure state on a C^*-algebra A.

Is ϕ multiplicative on some maximal abelian subalgebra of A? Aarnes and Kadison [1] have proved this for separable C^*-algebras. Another result along this line is

PROPOSITION 15. *Let A be a C^*-algebra with identity such that every irreducible representation of A contains the compact operators. If ϕ is a pure state of A, then there is a maximal abelian C^*-subalgebra B of A such that $\phi|B$ is multiplicative.*

PROOF. Let $A_0 = K(\phi) \cap K(\phi)^* = \{a \in A : \phi(a^*a) = \phi(aa^*) = 0\}$. Then A_0 and $\pi_\phi(A_0)$ are C^*-algebras, where π_ϕ is the irreducible representation induced by ϕ. We have $\pi_\phi(A_0) = \{\pi_\phi(a) : \pi_\phi(a)\xi_\phi = \pi_\phi(a^*)\xi_\phi = 0, a \in A\}$. Let $[\xi_\phi]$ be the one-dimensional subspace spanned by ξ_ϕ. If $[\xi_\phi] = H_\phi$, then π_ϕ is one-dimensional, ϕ is multiplicative and hence we may take B to be any maximal abelian C^*-subalgebra of A. So we may assume $[\xi_\phi]^\perp \neq 0$. Let $x \in B(H_\phi)$ be the projection onto $[\xi_\phi]^\perp$. Then $1-x$ is a compact operator on H_ϕ, and hence by hypothesis $1-x$ is in $\pi_\phi(A)$. So $x \in \pi_\phi(A)$ and $x = \pi_\phi(a_0)$ for some $a_0 \in A$, and since $\pi_\phi((a_0 + a_0^*)/2) = x$ we may assume that a_0 is selfadjoint. Then $\pi_\phi(a_0)\xi_\phi = 0$, so $a_0 \in A_0$. Then let B_0 be a maximal abelian C^*-subalgebra of A_0 containing a_0. Let $B = B_0 + \mathbf{C}e$, where e is the identity, then B is a C^*-subalgebra of A and ϕ is multiplicative on B since $\phi|B_0 = 0$. We show that B is maximal abelian in A. Let d be a selfadjoint element of A which commutes with B. Then $\pi_\phi(d)$ commutes with $\pi_\phi(a_0) = x$. So $[\xi_\phi]^\perp$ reduces $\pi_\phi(d)$ and $\pi_\phi(d)([\xi_\phi]) \subset [\xi_\phi]$. So $\pi_\phi(d)\xi_\phi = \delta\xi_\phi$ for some scalar δ, and δ is real since d is selfadjoint. Then $\pi_{\phi 1}(d - \delta e)\xi_\phi = 0$, so $d - \delta e \in A_0$. But $d - \delta e$ commutes with B_0, so $d - \delta e \in B_0$ since B_0 is maximal abelian in A_0, and hence $d \in B$. Thus B is maximal abelian in A.

ADDED IN PROOF. The conclusion of Proposition 15 remains true if ϕ is a pure state on any C^*-algebra A such that the associated representation π_ϕ is nonzero on the unique largest postliminal ideal of A (see [**3**, 4.3.6] for the definition of this ideal).

REFERENCES

1. John F. Aarnes and Richard V. Kadison, *Pure states and approximate identities*, Proc. Amer. Math. Soc. **21** (1969), 749–752.

2. A. Brown, P. R. Halmos and A. L. Shields, *Cesàro operators*, Acta Sci. Math. Szeged **26** (1965), 125–137. MR **32** #4539.

3. J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

4. R. G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415. MR **34** #3315.

5. L. Terrell Gardner, *On isomorphisms of C^*-algebras*, Amer. J. Math. **87** (1965), 384–396. MR **31** #3883.

6. Paul R. Halmos, *Introduction to Hilbert space*, Chelsea, New York, 1957.

7. ———, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.

8. R. V. Kadison and I. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400. MR **23** #A1243.

9. Bela Sz.-Nagy and Ciprian Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris and Akadémiai Kiadó, Budapest, 1967. MR **37** #778.

UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104