CHARACTERS ON SINGLY GENERATED 
C*-ALGEBRAS

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Abstract. In this note we consider the question of what elements $\delta$ in the spectrum of a bounded operator $A$ on Hilbert space have the property that there is a multiplicative linear functional $\phi$ on the C*-algebra generated by $A$ and $I$ whose value at $A$ is $\delta$. If $A$ is hyponormal then there is a character $\phi$ on the C*-algebra generated by $A$ and $I$ such that $\phi(A) = \delta$ if and only if $\delta$ is in the approximate point spectrum of $A$. We use this to prove a structure theorem for the C*-algebra generated by a hyponormal operator. We conclude by proving that any pure state on a Type I C*-algebra is multiplicative on some maximal abelian C*-subalgebra.

Let $A$ be an operator in $B(H)$, the set of bounded operators on a Hilbert space $H$. Let $C^*(A)$ be the C*-subalgebra of $B(H)$ generated by $A$ and the identity. By a character on a C*-algebra we mean a multiplicative linear functional. We will investigate the existence of characters on $C^*(A)$. Let $\text{sp}(A)$ denote the spectrum of $A$. The author began this research after learning that William Arveson had proved the following: If $\delta \in \text{sp}(A)$ is such that $\|A\| = |\delta|$, then there is a character $\phi$ on $C^*(A)$ such that $\phi(A) = \delta$. We independently prove this result at the end of this note. We denote by $a(A)$ the approximate point spectrum of $A$; i.e., $a(A)$ is the set of scalars $\delta$ for which there is a sequence $(x_n)$ of unit vectors in $H$ such that $(A - \delta I)x_n$ converges to zero in norm. Let $p(A)$ denote the set of eigenvalues of $A$. See [6, §31] for facts about spectra.

Proposition 1. Let $A \in B(H)$. Then

$$a(A) = \{ \delta \in \text{sp}(A) : C^*(A)(A - \delta I) \neq C^*(A) \}.$$  

Proof. First let $\delta \in a(A)$, then $A - \delta I$ is not bounded below. If $I$ were in $C^*(A)(A - \delta I)$, then we would have $I = D(A - \delta I)$ for some
\[ D \in C^*(A), \text{ and for } x \in H, \|x\| \leq \|D\| \|(A - \delta I)x\|; \text{ hence } A - \delta I \text{ would be bounded below. Thus } I \in C^*(A)(A - \delta I). \text{ Conversely, suppose } \delta \text{ is such that } C^*(A)(A - \delta I) \neq C^*(A). \text{ Then there is a pure state } \phi \text{ on } C^*(A) \text{ such that } C^*(A)(A - \delta I) \text{ is contained in } K(\phi), \text{ where } K(\phi) = \{ B \in C^*(A) : \phi(B^*B) = 0 \}, [3, 2.9.5]. \text{ If we had } \delta \notin a(A), \text{ then } A - \delta I \text{ would be bounded below and there would exist } m > 0 \text{ such that } m^2I \leq (A - \delta I)^*(A - \delta I). \text{ But } \phi((A - \delta I)^*(A - \delta I)) = 0, \text{ so } m^2\phi(I) = 0, \text{ but this is a contradiction. Hence } \delta \in a(A). \]

The next two corollaries say roughly that to compute the approximate point spectrum of an element we can use any \( C^* \)-algebra containing the element.

**Corollary 2.** Let \( \delta \in \text{sp}(A) \). Then if \( I \in B(H)(A - \delta I) \), we also have \( I \in C^*(A)(A - \delta I) \).

**Proof.** If \( I = D(A - \delta I) \) for some \( D \in B(H) \), then as above \( (A - \delta I) \) is bounded below so \( \delta \notin a(A) \), and by the proposition \( C^*(A)(A - \delta I) = C^*(A) \).

**Corollary 3.** Let \( A \in B(H) \). Then

\[ a(A) = \{ \delta \in \text{sp}(A) : B(H)(A - \delta I) \neq B(H) \}. \]

For facts about irreducible representations see [3, §2].

**Corollary 4.** If \( \delta \in a(A) \), then there exists a pure state \( \phi \) on \( C^*(A) \) such that \( \pi_\phi(A)\xi_\phi = \delta \xi_\phi \), where \( \pi_\phi \) is the irreducible representation induced by \( \phi \), with canonical cyclic vector \( \xi_\phi \).

**Proof.** If \( \delta \in a(A) \), then \( C^*(A)(A - \delta I) \neq C^*(A) \); so there is a pure state \( \phi \) on \( C^*(A) \) such that \( C^*(A)(A - \delta I) \subset K(\phi) \). Then \( A + K(\phi) = \delta I + K(\phi) \), so \( \pi_\phi(A)\xi_\phi = \delta \xi_\phi \).

**Corollary 5.** If \( \delta \in a(A) \), then \( \delta \in a(\pi(A)) \) for any representation \( \pi \) of \( C^*(A) \) on a Hilbert space. (We assume \( \pi(I) = I \).)

**Proof.** If \( \delta \in a(A) \), then \( I = D(A - \delta I) \) for some \( D \in C^*(A) \), so \( \pi(I) = I = \pi(D)(\pi(A) - \delta I) \). Hence \( \delta \in a(\pi(A)) \).

For facts about the universal representation see [3, 2.7.8].

**Corollary 6.** Let \( \pi_0 : C^*(A) \to B(H_0) \) be the universal representation of \( C^*(A) \). Then \( a(A) = p(\pi_0(A)) = a(\pi_0(A)) \) and \( a(A^*) = p(\pi_0(A^*)) = a(\pi_0(A^*)) \).

**Proof.** First note that by Proposition 1, \( a(A) \) depends only on \( C^*(A) \) and hence \( a(A) = a(\pi_0(A)) \). But by Corollary 4, \( a(A) \subset p(\pi_0(A)) \).
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\[ a(\pi_0(A)) = a(A). \] So \( a(A) = \rho(\pi_0(A)) = a(\pi_0(A)). \) Since \( C^*(A) = C^*(A^*) \), we also have \( a(A^*) = \rho(\pi_0(A^*)) = a(\pi_0(A^*)). \)

The conclusion of Corollary 6 holds if we take for \( \pi_0 \) the atomic representation of \( C^*(A) \); i.e., the direct sum of all irreducible representations [5, p. 388].

**Proposition 7.** Let \( A \in B(H) \) and \( \delta \in \text{sp}(A) \). Then there is a pure state \( \phi \) on \( C^*(A) \) such that \( \phi(A) = \delta. \)

**Proof.** First suppose \( \delta \in a(A) \) and let \( \phi \) be a pure state on \( C^*(A) \) such that \( C^*(A)(A - \delta I) \subseteq K(\phi) \). Then \( \phi(A) = \delta. \) If \( \delta \notin a(A) \), then \( A - \delta I \) does not have dense range and \( A* - \delta I \) is not one-to-one. So \( \delta \notin a(A^*) \) and as above there is a pure state \( \phi \) on \( C^*(A) \) such that \( \phi(A^*) = \delta \), or \( \phi(A) = \delta. \)

**Proposition 8.** If \( A \in B(H) \) and \( \phi \) is a character on \( C^*(A) \), then \( \phi(A) \in a(A). \)

**Proof.** Clearly \( A - \phi(A)I \subseteq \phi^{-1}(0) \), which is a proper two-sided ideal of \( C^*(A) \). So \( \phi(A) \in \text{sp}(A) \). Also \( C^*(A)(A - \phi(A)I) \subseteq \phi^{-1}(0) \neq C^*(A) \), so by Proposition 1, \( \phi(A) \in a(A). \)

**Proposition 9.** Let \( A \in B(H) \) and let \( \delta \in a(A) \) be such that \( (A - \delta I) \cdot (A^* - \bar{\delta}I) \subseteq N_2(A^* - \bar{\delta}I)(A - \delta I) \) for some \( N > 0 \). Then there exists a character \( \phi \) on \( C^*(A) \) such that \( \phi(A) = \delta. \)

**Proof.** By the Radon-Nikodým theorem for operators [4], there exists an operator \( D \in B(H) \) such that \( A - \delta I = (A^* - \bar{\delta}I)D \), or \( A^* - \bar{\delta}I = D^*(A - \delta I) \). Hence \( B(H)(A^* - \bar{\delta}I) \subseteq B(H)(A - \delta I) \subseteq K(\phi) \). Then for all \( B \in B(H) \) we have \( \phi(BA^*) = \phi(B)\bar{\delta} = \phi(B)\phi(A^*) \) and \( \phi(BA) = \phi(B)\phi(A) \). Hence if \( M(A, A^*) \) is any monomial in \( A \) and \( A^* \), we have that \( \phi(M(A, A^*)) = M(\phi(A), \phi(A^*)) \). Hence \( \phi \) is a character on \( C^*(A) \) and \( \phi(A) = \delta. \)

An operator \( A \) on \( H \) is called *hyponormal* if \( AA^* \leq A^*A \). If \( A \) is hyponormal, then \( A - \delta I \) is also hyponormal for all scalars \( \delta \). Then as an immediate corollary of Proposition 9 we have

**Corollary 10.** Let \( A \in B(H) \) be hyponormal. Then for all \( \delta \in a(A) \), there exists a character \( \phi \) on \( C^*(A) \) such that \( \phi(A) = \delta. \)

**Corollary 11.** Let \( A \in B(H) \) be hyponormal. Let \( S \) be the set of all nonzero characters on \( C^*(A) \). Let \( I = \cap \{ \phi^{-1}(0) : \phi \in S \} \). Then \( C^*(A)/I \) is isomorphic to the \( C^* \)-algebra of continuous complex-valued functions on \( a(A) \).
Proof. By Corollary 10, \( S \) is nonempty. So \( I \) is a closed two-sided ideal, and \( C^*(A)/I \) is a \( C^* \)-algebra with identity which is clearly commutative. So \( C^*(A)/I \) is isomorphic to \( C(X) \) for some compact Hausdorff space \( X \). But \( C^*(A)/I \) is singly generated by the normal element \( A + I \), hence \( X = \text{spec}(A + I) = \{ \delta \in \phi : \phi \text{ is a character of } C^*(A)/I \} \). Let \( \pi : C^*(A) \rightarrow C^*(A)/I \) be the natural projection. Then clearly the map \( \phi \rightarrow \phi \circ \pi \) is a bijection between the set of characters on \( C^*(A)/I \) and the set \( S \) of characters on \( C^*(A) \). So \( X = \{ \phi(A) : \phi \in S \} \). Then by Corollary 10, \( a(A) \subseteq X \) and by Proposition 8, \( X \subseteq a(A) \). So \( X = a(A) \).

In certain cases, Corollary 10 may be used to compute the approximate point spectrum of an operator. For example, consider the discrete Cesàro operator on \( l^2 \) [2], \( C_0 \), given by

\[
(C_0 x)(n) = 1/(n + 1)(x_0 + x_1 + \cdots + x_n), \quad \text{for } n = 0, 1, 2, \ldots ,
\]

where \( x = (x_0, x_1, x_2, \cdots ) \) is in \( l^2 \). Brown, Halmos, and Shields [2] proved that \( \|C_0\| = 2 \), \( \text{spec}(C_0) = \{ \delta : |1 - \delta| \leq 1 \} \), the point spectrum of \( C_0 \) is empty, and \( C_0 \) is hyponormal. Computing the approximate point spectrum of \( C_0 \) directly seems somewhat difficult, but using Corollary 10 we can show that \( a(C_0) = \{ \delta : |1 - \delta| = 1 \} \). If \( |1 - \delta| = 1 \), then \( \delta \) is in the boundary of \( \text{spec}(C_0) \), so \( \delta \in a(C_0) \) [7, Problem 63]. Conversely let \( \delta \in a(C_0) \), then there is a character \( \phi \) on \( C^*(C_0) \) such that \( \phi(C_0) = \delta \). Now it is known and easily shown that \( C_0 + C_0^* - C_0 C_0^* = D_0 \) is a diagonal operator with diagonal 1, 1/2, 1/3, \ldots and is hence compact. Now if an operator \( A \) commutes with \( C_0 \) and \( C_0^* \), then \( A \) commutes with \( D_0 \) and is thus a diagonal operator. Also, it is easily seen that the only diagonal operator commuting with \( C_0 \) is a scalar, so \( A \) is a scalar, and \( C_0 \) is hence irreducible. Thus by [3, 4.1.10] \( C^*(C_0) \) contains the compact operators on \( l^2 \). But then \( \phi \) is a character on the simple \( C^* \)-algebra of compact operators on \( l^2 \), so \( \phi \) vanishes on the compacts. Thus

\[
\phi(C_0) + \overline{\phi(C_0)} - |\phi(C_0)|^2 = \phi(D_0) = 0.
\]

If \( \phi(C_0) \) has real part \( a \) and imaginary part \( b \), this becomes \( 2a - a^2 - b^2 = 0 \), or \( (a - 1)^2 + b^2 = 1 \). Hence \( \phi(C_0) = \delta \) is on the circle of radius 1 and center \( (1, 0) \), so \( |1 - \delta| = 1 \). So \( a(C_0) = \{ \delta : |1 - \delta| = 1 \} \).

We remark that the converse of Proposition 9 is not true: Let \( (e_n)^\infty \) be the canonical orthonormal basis in \( l^2 \) and define a weighted shift \( D \in B(l^2) \) by \( D e_n = \delta_n e_{n+1} \), where \( \delta_{2n} = (1/2)^{2n} \) for \( n \geq 0 \) and \( \delta_{2n+1} = (1/3)^{2n+1} \) for \( n \geq 0 \). Then \( D^*(x_0, x_1, \cdots ) = (\delta_0 x_1, \delta_1 x_2, \delta_2 x_3, \cdots ) \) and \( DD^*(x_0, x_1, \cdots ) = (0, \delta_0^2 x_1, \delta_1^2 x_2, \cdots ) \), \( D^* D(x_0, x_1, \cdots ) = \cdots \)
(δ^j_0 x_0, δ^j_1 x_1, \ldots ). Suppose there existed an \( N > 0 \) such that \( DD^* \leq N^2 D^* D \). Then \( \delta^j_0 \leq N^2 \delta^j_{n+1} \) for all \( j \geq 0 \). But \( \frac{\delta^j_n}{\delta^j_{n+1}} = (3/2)^{2n} 3 \), which is unbounded. So there does not exist \( N > 0 \) such that \( DD^* \leq N^2 D^* D \). If \( D^* D \leq N^2 D D^* \) then \( \| D x \| \leq N \| D^* x \| \) for all \( x \in l^2 \). But \( D^* e_0 = 0 \) while \( D e_0 \neq 0 \). So there does not exist \( N > 0 \) such that \( D^* D \leq N^2 D D^* \). Now \( DD^* \) is clearly compact, so \( D \) is compact. By [7, Problem 151] \( D \) is irreducible, so by [3, 4.1.10] \( C^*(D) \) is just the scalars plus the compacts. The projection \( \pi \) of \( C^*(D) \) onto \( C^*(D) \) modulo the compacts is then a character and \( \pi(D) = 0 \). This shows that the converse of Proposition 9 is true for neither \( D \) nor \( D^* \).

To prove Arveson’s result (Proposition 13 and Corollary 14) we need the following lemma [9, p. 8].

**Lemma 12.** If \( T \in B(H) \) is such that \( \| T \| \leq 1 \) and \( x \in H \) is such that \( Tx = x \), then \( T^* x = x \).

**Proof.** Since \( Tx = x \), we have \( (x, T^* x) = (Tx, x) = \| x \|^2 \). Hence

\[
\| x - T^* x \|^2 = \| x \|^2 - (x, T^* x) - (T^* x, x) + \| T^* x \|^2 \\
= \| x \|^2 - 2 \text{ Re} (x, T^* x) + \| T^* x \|^2 = \| x \|^2 - 2 \| x \|^2 + \| T^* x \|^2 \\
= \| T^* x \|^2 - \| x \|^2 \leq 0.
\]

So \( x = T^* x \).

**Proposition 13.** Let \( A \in B(H) \) be such that \( \| A \| \leq 1 \) and suppose \( 1 \in \text{sp}(A) \). Then there exists a character \( \phi \) on \( C^*(A) \) such that \( \phi(A) = 1 \).

**Proof.** Since \( 1 \) is in the boundary of \( \text{sp}(A) \), we have \( 1 \in a(A) \). Let \( \pi_0 : C^*(A) \to B(H_0) \) be the universal representation. Then by Corollary 6 we have \( a(A) = \rho(\pi_0(A)) = a(\pi_0(A)) \). Hence \( 1 \in \rho(\pi_0(A)) \) and there is an \( x \in H_0 \) such that \( \pi_0(A)x = x \). Then by the lemma \( \pi_0(A^*)x = x \). Then if \( \omega_x \) is the vector state associated with \( x \), we have for \( M(\pi_0(A), \pi_0(A^*)) \) any monomial in \( \pi_0(A) \) and \( \pi_0(A^*) \),

\[
\omega_x(M(\pi_0(A), \pi_0(A^*))) = M(1, 1).
\]

Hence \( \omega_x \) is a character on \( \pi_0(C^*(A)) \), so \( \omega_x \circ \pi_0 \) is a character on \( C^*(A) \) and \( \omega_x \circ \pi_0(A) = \| x \|^2 = 1 \).

**Corollary 14.** Let \( A \) be a nonzero element of \( B(H) \) and suppose there is a \( \delta \in \text{sp}(A) \) such that \( \| A \| = | \delta | \). Then there is a character \( \phi \) on \( C^*(A) \) such that \( \phi(A) = \delta \).

**Proof.** Consider \( B = \delta^{-1} A \) and use Proposition 13.

We briefly consider an unrelated \( C^* \)-algebra problem first raised by Kadison and Singer [8]. Let \( \phi \) be a pure state on a \( C^* \)-algebra \( A \).
Is \( \phi \) multiplicative on some maximal abelian subalgebra of \( A \)? Aarnes and Kadison [1] have proved this for separable \( C^* \)-algebras. Another result along this line is

**Proposition 15.** Let \( A \) be a \( C^* \)-algebra with identity such that every irreducible representation of \( A \) contains the compact operators. If \( \phi \) is a pure state of \( A \), then there is a maximal abelian \( C^* \)-subalgebra \( B \) of \( A \) such that \( \phi \mid B \) is multiplicative.

**Proof.** Let \( A_0 = K(\phi) \cap K(\phi)^* = \{ a \in A : \phi(a^*a) = \phi(aa^*) = 0 \} \). Then \( A_0 \) and \( \pi_\phi(A_0) \) are \( C^* \)-algebras, where \( \pi_\phi \) is the irreducible representation induced by \( \phi \). We have \( \pi_\phi(A_0) = \{ \pi_\phi(a) : \pi_\phi(a) \xi_\phi = \pi_\phi(a^*) \xi_\phi = 0, a \in A \} \). Let \( [\xi_\phi] \) be the one-dimensional subspace spanned by \( \xi_\phi \). If \( [\xi_\phi] = H_\phi \), then \( \pi_\phi \) is one-dimensional, \( \phi \) is multiplicative and hence we may take \( B \) to be any maximal abelian \( C^* \)-subalgebra of \( A \). So we may assume \( [\xi_\phi] \perp \perp 0 \). Let \( x \in B(H_\phi) \) be the projection onto \( [\xi_\phi] \perp \perp \). Then \( 1 - x = 0 \) is a compact operator on \( H_\phi \), and hence by hypothesis \( 1 - x \) is in \( \pi_\phi(A) \). So \( x \in \pi_\phi(A) \) and \( x = \pi_\phi(a_0) \) for some \( a_0 \in A \), and since \( \pi_\phi((a_0 + a_0^*)/2) = x \) we may assume that \( a_0 \) is selfadjoint. Then \( \pi_\phi(a_0) \xi_\phi = 0 \), so \( a_0 \in A_0 \). Then let \( B_0 \) be a maximal abelian \( C^* \)-subalgebra of \( A_0 \) containing \( a_0 \). Let \( B = B_0 + Ce \), where \( e \) is the identity, then \( B \) is a \( C^* \)-subalgebra of \( A \) and \( \phi \) is multiplicative on \( B \) since \( \phi \mid B_0 = 0 \). We show that \( B \) is maximal abelian in \( A \). Let \( d \) be a selfadjoint element of \( A \) which commutes with \( B \). Then \( \pi_\phi(d) \) commutes with \( \pi_\phi(a_0) \). So \( [\xi_\phi] \perp \perp \) reduces \( \pi_\phi(d) \) and \( \pi_\phi(d)([\xi_\phi]) \subset [\xi_\phi] \). So \( \pi_\phi(d) \xi_\phi = \delta \xi_\phi \) for some scalar \( \delta \), and \( \delta \) is real since \( d \) is selfadjoint. Then \( \pi_\phi(d - \delta e) \xi_\phi = 0 \), so \( d - \delta e \in A_0 \). But \( d - \delta e \) commutes with \( B_0 \), so \( d - \delta e \in B_0 \) since \( B_0 \) is maximal abelian in \( A_0 \), and hence \( d \in B \). Thus \( B \) is maximal abelian in \( A \).

**Added in proof.** The conclusion of Proposition 15 remains true if \( \phi \) is a pure state on any \( C^* \)-algebra \( A \) such that the associated representation \( \pi_\phi \) is nonzero on the unique largest postliminal ideal of \( A \) (see [3, 4.3.6] for the definition of this ideal).

**References**


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