RINGS OF CONTINUOUS FUNCTIONS AND TOTALLY INTEGRALLY CLOSED RINGS

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0. Introduction. We shall prove that the ring of continuous complex-valued functions on a completely regular Hausdorff space is totally integrally closed iff the space is extremally disconnected.

$X$ always denotes a completely regular Hausdorff space. $R$ is the field of real numbers, $C$ of complex numbers. $C(X)$ denotes the ring of all continuous $R$-valued functions on $X$, $C^*(X)$ the subring of bounded functions in $C(X)$. $i = \sqrt{-1}$. Then $C(X)[i]$ (respectively, $C^*(X)[i]$) is the ring of continuous (respectively, bounded continuous) $C$-valued functions of $X$.

A ring $A$ (ring $\equiv$ commutative ring with 1) is totally integrally closed if for every integral extension $h: B \to C$, the induced map $h^*: \text{Hom}(C, A) \to \text{Hom}(B, A)$ is surjective. See [1] and [4] for a detailed treatment. By [4, Theorem 1 and Proposition 5], $A$ is totally integrally closed iff the following four conditions hold:

(1) $A$ is reduced,
(2) every monic polynomial of positive degree in one indeterminate over $A$ factors completely into monic linear factors,
(3) distinct minimal primes of $A$ are comaximal, and
(4) the set of minimal primes of $A$, $\text{Min Spec } A$, is an extremally disconnected space in the inherited Zariski topology.

1. The result.

Theorem. The following conditions on a completely regular Hausdorff space $X$ are equivalent.

(a) $X$ is extremally disconnected.
(b) $C(X)[i]$ is totally integrally closed.
(c) $C^*(X)[i]$ is totally integrally closed.

Proof. We show (b)$\iff$(c)$\iff$(a). Let $S$ be the set of functions in $C^*(X)[i]$ which never vanish. It is easy to see that $C(X)[i] = S^{-1}C^*(X)[i]$, and that $C^*(X)[i]$ is integrally closed in $C(X)[i]$. By [4, Propositions 1 and 7], these two rings are totally integrally closed or not alike. Now assume (c). The obvious map $X \to \text{Spec } C^*(X)[i]$ which takes $x$ into $\{f: f(x) = 0\}$ is an embedding, and

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the image is dense. But Spec $C^*(X)[i]$ must be extremal [4, Proposition 3]; hence, $X$ must be extremally disconnected.

Now assume (a). We verify conditions (1)-(4) given in the introduction for $A = C^*(X)[i]$. $A$ is obviously reduced (no nilpotents). Condition (3) is immediate because $X$ is extremally disconnected $\Rightarrow X$ is an $F$-space [2, p. 214, N4], and $X$ is an $F$-space $\iff$ (3) holds [2, p. 208]. (4) is immediate from [3, Theorem 5.3]. I wish to thank the referee for this short proof of (3) and (4). To prove (2), we first note that $X$ is zero dimensional in the sense of [2, p. 246]. (A space $X$ is zero dimensional iff, equivalently, either (1) disjoint zero sets in $X$ have disjoint clopen neighborhoods, or (2) the Stone-Cech compactification $\beta X$ of $X$ has a clopen basis.) $X$ is zero dimensional iff $\beta X$ is, and if $X$ is zero dimensional, so is each cozero set $U \subseteq X$. For if $f$ is a nonnegative function vanishing precisely on $X-U$, for each positive integer $n$, let $V_n$ be a clopen set in $X$ containing the zero set $f^{-1}([0, 1/n, +\infty])$ and disjoint from the zero set $f^{-1}([0, 1/(n+1)])$: then $V_n \subseteq V_{n+1}$, all $n$, and $U = \bigcup_n (V_{n+1} - V_n)$ is a disjoint union of clopen subsets of the zero dimensional space $X$. We are now ready to complete the proof of the theorem by establishing

**Lemma.** If $X$ is zero dimensional, then every monic polynomial of positive degree in one indeterminate $t$ over $C^*(X)[i]$ splits into monic linear factors.

**Proof.** Let $F = F(t) = t^n + \sum_{j=0}^{n-1} h_j t^j \in C^*(X)[i][t]$ be monic of degree $n \geq 1$. For each $x$ in $X$ let $F_x(t) = t^n + \sum_{j=0}^{n-1} h_j(x) t^j \in C[t]$. We use induction on $n$. The case $n=1$ is trivial. Assume that $n>1$ and that smaller degree polynomials split (for all zero dimensional $X$). Let $Y = \{y \in X: \text{all roots of } F_y \text{ are equal, i.e. } F_y = (t - c_y)^n \text{ for some } c_y \in C\}$. $\sum_{j=0}^{n-2} |h_j - (\zeta)(h_{n-1}/n)^{n-j}|$ vanishes precisely on $Y$. Hence $X - Y$ is a clopen set and is zero dimensional. Suppose $F \mid X - Y$ splits. Then we can extend each root function continuously (uniquely) to $X$ by giving it the value $c_y$ for each $y \in Y$. Continuity here follows from the Hurwitz theorem [5, p. 4, Theorem (1.5)]. Hence, we may assume without loss of generality that $Y = \emptyset$.

By considering $F(t - (1/n)h_{n-1})$ instead of $F$ we may also assume that $h_{n-1} = 0$. Let $h = \sum_{j=0}^{n-2} |h_j|^{1/n-j} \in C^*(X)$. Since $Y = \emptyset$, the roots of $F_x$ are never all zero, and hence $h$ never vanishes. Let $G = t^n + \sum_{j=0}^{n-1} g_j t^j = h^{-n} F(ht)$, so that $g_j = h_j/h^{n-j}$ (in particular, $g_{n-1} = 0$). An easy computation shows $|g_j| \leq 1$ for each $j$, so that $G \in C^*(X)[i][t]$, and it is clear that if $G$ splits, so does $F$. Since $\sum_{j=0}^{n-2} |g_j|^{1/n-j} = 1$, we may also assume without loss of generality that $h = 1$. 
We use a superscript \( e \) to denote the unique continuous extension of an element of \( C^*(X)[i] \) to \( \beta X \), and also to denote the result of extending all the coefficients of a polynomial. Clearly, it suffices to show that \( F^e \subseteq C^*(\beta X)[i][t] \) splits. We still have \( h_{n-1}^e = 0 \). Moreover, since \( h = 1 \), \( H^e = \sum_{j=0}^{n-2} |h_j|^j/n-j = 1 \), so that the coefficients of \( F^e \) never vanish simultaneously. Since \( h_{n-1}^e = 0 \), the roots of \( F^e \) are never all equal, for then they would all have to be zero. Thus, the hypothesis of \( Y = \emptyset \) is preserved if we pass to consideration of \( \beta X \) instead of \( X \).

Thus, we may assume without loss of generality that \( X \) is compact and that \( F_x \) does not have \( n \) equal roots at any point. It now suffices to show that each point \( x \in X \) has some neighborhood \( U \) such that \( F|U \) splits over \( U \), for we then get a finite cover of \( X \) by clopen sets on which \( F \) splits. Let \( x \in X \) be arbitrary. Partition the roots of \( F_x \) into two disjoint nonempty sets \( A, B \), and choose \( \epsilon > 0 \) less than half the distance of \( A \) from \( B \). For some sufficiently small open neighborhood \( U \) of \( x \), for each \( u \in U \) there will be a corresponding partition \( A(u), B(u) \) of the roots of \( F_u \), where, counting multiplicities, \( A(u) \) has the same number of elements as \( A \) and \( B(u) \) as \( B \). This follows from the Hurwitz theorem: \( A(u) \) consists of those roots within distance \( \epsilon \) of some element of \( A \), and similarly for \( B(u) \). Let

\[
G_u = \prod_{a \in A(u)} (t - a)^{m(a)}
\]

and

\[
H_u = \prod_{b \in B(u)} (t - b)^{m(b)}
\]

where \( m(c) \) is the multiplicity of \( c \) as a root of \( F_u \). It follows easily from the Hurwitz theorem that the coefficients of \( G_u \) and \( H_u \) depend continuously on \( u \), so that \( F \) factors nontrivially \( F = GH \) over \( U \). By the induction hypothesis, \( G \) and \( H \) both split over \( U \).

Note that the conclusion of the lemma fails if we let \( X = \mathbb{C} \) or \( X = \{ c \in \mathbb{C} : |c| = 1 \} \). Let \( h \) be the inclusion map of \( X \) into \( \mathbb{C} \), and let \( F(t) = t^n - h \) for any integer \( n \geq 2 \). The referee has pointed out the following startling example where \( X \) has a basis of clopen sets but the lemma fails: In the construction of the space \( \Delta_1 \) in 16M on p. 264 of [2], use the unit circle of the complex plane instead of \([0, 1]\), and let \( h \) be the restricted product projection from the resulting space \( X \) to the circle. Then once more \( F(t) = t^n - h, n \geq 2 \), does not split.
ADDED IN PROOF. The lemma follows from the results of R. S. Countryman, *On the characterization of compact Hausdorff X for which C(X) is algebraically closed*, Pacific J. Math. 20 (1967), 433–448.

REFERENCES


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