RINGS OF CONTINUOUS FUNCTIONS AND TOTALLY INTEGRALLY CLOSED RINGS[1]

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0. **Introduction.** We shall prove that the ring of continuous complex-valued functions on a completely regular Hausdorff space is totally integrally closed iff the space is extremally disconnected.

X always denotes a completely regular Hausdorff space. **R** is the field of real numbers, **C** of complex numbers. $C(X)$ denotes the ring of all continuous **R**-valued functions on X, $C^*(X)$ the subring of bounded functions in $C(X)$. $i=\sqrt{(-1)}$. Then $C(X)[i]$ (respectively, $C^*(X)[i]$) is the ring of continuous (respectively, bounded continuous) **C**-valued functions of X.

A ring A (ring $\equiv$ commutative ring with 1) is *totally integrally closed* if for every integral extension $h: B \to C$, the induced map $h^*: \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ is surjective. See [**1**] and [**4**] for a detailed treatment. By [**4**, Theorem 1 and Proposition 5], A is totally integrally closed iff the following four conditions hold:

(1) A is reduced,

(2) every monic polynomial of positive degree in one indeterminate over A factors completely into monic linear factors,

(3) distinct minimal primes of A are comaximal, and

(4) the set of minimal primes of A, Min Spec A, is an extremally disconnected space in the inherited Zariski topology.

1. **The result.**

Theorem. *The following conditions on a completely regular Hausdorff space X are equivalent.*

(a) *X is extremally disconnected.*

(b) *$C(X)[i]$ is totally integrally closed.*

(c) *$C^*(X)[i]$ is totally integrally closed.*

Proof. We show (b)$\Leftrightarrow$(c)$\Leftrightarrow$(a). Let S be the set of functions in $C^*(X)[i]$ which never vanish. It is easy to see that $C(X)[i] = S^{-1}C^*(X)[i]$, and that $C^*(X)[i]$ is integrally closed in $C(X)[i]$. By [**4**, Propositions 1 and 7], these two rings are totally integrally closed or not alike. Now assume (c). The obvious map $X \to$ Spec $C^*(X)[i]$ which takes x into $\{f: f(x)=0\}$ is an embedding, and

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the image is dense. But Spec $C^*(X)[i]$ must be extremal [**4**, Proposition 3]; hence, X must be extremally disconnected.

Now assume (a). We verify conditions (1)–(4) given in the introduction for $A=C^*(X)[i]$. A is obviously reduced (no nilpotents). Condition (3) is immediate because X is extremally disconnected $\Rightarrow X$ is an F-space [**2**, p. 214, N4], and X is an F-space $\Leftrightarrow$(3) holds [**2**, p. 208]. (4) is immediate from [**3**, Theorem 5.3]. I wish to thank the referee for this short proof of (3) and (4). To prove (2), we first note that X is zero dimensional in the sense of [**2**, p. 246]. (A space X is zero dimensional iff, equivalently, either (1) disjoint zero sets in X have disjoint clopen neighborhoods, or (2) the Stone-Čech compactification βX of X has a clopen basis.) X is zero dimensional iff βX is, and if X is zero dimensional, so is each cozero set $U\subset X$. For if f is a nonnegative function vanishing precisely on $X-U$, for each positive integer n, let V_n be a clopen set in X containing the zero set $f^{-1}([1/n, +\infty])$ and disjoint from the zero set $f^{-1}([0, 1/(n+1)])$: then $V_n\subset V_{n+1}$, all n, and $U=\bigcup_n(V_{n+1}-V_n)$ is a disjoint union of clopen subsets of the zero dimensional space X. We are now ready to complete the proof of the theorem by establishing

LEMMA. *If X is zero dimensional, then every monic polynomial of positive degree in one indeterminate t over $C^*(X)[i]$ splits into monic linear factors.*

PROOF. Let $F=F(t)=t^n+\sum_{j=0}^{n-1} h_jt^j\in C^*(X)[i][t]$ be monic of degree $n\geqq 1$. For each x in X let $F_x(t)=t^n+\sum_{j=0}^{n-1} h_j(x)t^j\in\mathbf{C}[t]$. We use induction on n. The case $n=1$ is trivial. Assume that $n>1$ and that smaller degree polynomials split (for all zero dimensional X). Let $Y=\{y\in X$: all roots of F_y are equal, i.e. $F_y=(t-c_y)^n$ for some $c_y\in C\}$. $\sum_{j=0}^{n-2}|h_j-\binom{n}{j}(h_{n-1}/n)^{n-j}|$ vanishes precisely on Y. Hence $X-Y$ is a cozero set and is zero dimensional. Suppose $F|X-Y$ splits. Then we can extend each root function continuously (uniquely) to X by giving it the value c_y for each $y\in Y$. Continuity here follows from the Hurwitz theorem [**5**, p. 4, Theorem (1.5)]. Hence, we may assume without loss of generality that $Y=\varnothing$.

By considering $F(t-(1/n)h_{n-1})$ instead of F we may also assume that $h_{n-1}=0$. Let $h=\sum_{j=0}^{n-2}|h_j|^{1/n-j}\in C^*(X)$. Since $Y=\varnothing$, the roots of F_x are never all zero, and hence h never vanishes. Let $G=t^n+\sum_{j=0}^{n-1} g_jt^j=h^{-n}F(ht)$, so that $g_j=h_j/h^{n-j}$ (in particular, $g_{n-1}=0$). An easy computation shows $|g_j|\leqq 1$ for each j, so that $G\in C^*(X)[i][t]$, and it is clear that if G splits, so does F. Since $\sum_{j=0}^{n-2}|g_j|^{1/n-j}=1$, we may also assume without loss of generality that $h=1$.

We use a superscript e to denote the unique continuous extension of an element of $C^*(X)[i]$ to βX, and also to denote the result of extending all the coefficients of a polynomial. Clearly, it suffices to show that $F^e \in C^*(\beta X)[i][t]$ splits. We still have $h^e_{n-1}=0$. Moreover, since $h=1$, $H^e=\sum_{j=0}^{n-2}|h^e_j|^{1/n-j}=1$, so that the coefficients of F^e never vanish simultaneously. Since $h^e_{n-1}=0$, the roots of F^l_x are never all equal, for then they would all have to be zero. Thus, the hypothesis of $Y=\emptyset$ is preserved if we pass to consideration of βX instead of X.

Thus, we may assume without loss of generality that X is compact and that F_x does not have n equal roots at any point. It now suffices to show that each point $x \in X$ has some neighborhood U such that $F|U$ splits over U, for we then get a finite cover of X by clopen sets on which F splits. Let $x \in X$ be arbitrary. Partition the roots of F_x into two disjoint nonempty sets A, B, and choose $\epsilon > 0$ less than half the distance of A from B. For some sufficiently small open neighborhood U of x, for each $u \in U$ there will be a corresponding partition $A(u)$, $B(u)$ of the roots of F_u, where, counting multiplicities, $A(u)$ has the same number of elements as A and $B(u)$ as B. This follows from the Hurwitz theorem: $A(u)$ consists of those roots within distance ϵ of some element of A, and similarly for $B(u)$. Let

$$G_u = \prod_{a \in A(u)} (t-a)^{m(a)}$$

and

$$H_u = \prod_{b \in B(u)} (t-b)^{m(b)}$$

where $m(c)$ is the multiplicity of c as a root of F_u. It follows easily from the Hurwitz theorem that the coefficients of G_u and H_u depend continuously on u, so that F factors nontrivially $F=GH$ over U. By the induction hypothesis, G and H both split over U.

Note that the conclusion of the lemma fails if we let $X=\mathbf{C}$ or $X=\{c \in \mathbf{C}: |c|=1\}$. Let h be the inclusion map of X into $\mathbf{C}$, and let $F(t)=t^n-h$ for any integer $n \geqq 2$. The referee has pointed out the following startling example where X has a basis of clopen sets but the lemma fails: In the construction of the space Δ_1 in 16M on p. 264 of [2], use the unit circle of the complex plane instead of $[0, 1]$, and let h be the restricted product projection from the resulting space X to the circle. Then once more $F(t)=t^n-h$, $n \geqq 2$, does not split.

Added in proof. The lemma follows from the results of R. S. Countryman, *On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed*, Pacific J. Math. **20** (1967), 433–448.

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