WALLMAN-TYPE COMPACTIFICATIONS

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Abstract. All spaces in this paper are Tychonoff. A Wallman base on a space $X$ is a normal separating ring of closed subsets of $X$ (see Steiner, Duke Math. J. 35 (1968), 269–276). Let $T$ be a compact space and $\mathcal{L}$ a Wallman base on $T$. For $X \subset T$, define $\mathcal{L}_X = \{A \cap X \mid A \in \mathcal{L}\}$.

Theorem 1. If $X$ is a dense subspace of $T$, then $T = w\mathcal{L}_X$ iff $\text{cl}_T A \cap \text{cl}_T B = \emptyset$ whenever $A, B \in \mathcal{L}_X$ and $A \cap B = \emptyset$.

Theorem 2. $T = w\mathcal{L}_X$ for each dense $X \subset T$ iff $T = w\mathcal{L}_Y$ for each dense $Y \subset T$ where $T \neq Y \in \mathcal{L}$.

From these theorems we show that every compact $F$-space and every compact orderable space is a Wallman compactification of each of its dense subspaces.

1. Introduction. The Wallman base concept, first initiated by H. Wallman [W] in 1938, has been successfully used by Frink [3], Alo and Shapiro [1], Brooks [2], Hager [4], E. Steiner and A. Steiner [8], [9], and others. An important reference is the text Rings of continuous functions [GJ] which exploits Wallman's method in constructing the Stone-Čech compactification. We shall rely on these references for background, definitions and terminology.

The chief motivation for this paper is an answer to the question: Can every Hausdorff compactification of a Tychonoff space be constructed by Wallman's method? We provide some partial solutions.

All topological spaces in this paper are Tychonoff. The space $T$ is a compactification of $X$ (denoted $T \subset \beta X$) means $T$ is a compact extension of $X$. When referring to an extension $T$ of $X$, we may (and do) take $X$ as a subspace of $T$ (tacitly assuming the necessary embedding maps). Accordingly, we shall write $T_1 = T_2$ when $T_1$ and $T_2$ are equivalent compactifications of a given space. The notation $T \subset wX$ ($T \subset zX$) means $T$ is a Wallman compactification ($z$-compactification) of $X$. We reserve $\beta X$ for the Stone-Čech compactification of $X$. Finally, $\text{cl}_X A$ and $\text{int}_X A$ denote the closure and interior, respectively, of $A$ in $X$.

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2. Wallman bases on compact spaces. The collection \( \mathcal{L} \) of subsets of \( X \) is a lattice on \( X \) means that \( \emptyset, X \in \mathcal{L} \) and \( \mathcal{L} \) is closed under finite intersections and unions [2]. We first provide necessary and sufficient conditions for a lattice on a compact space \( T \) to be a Wallman base on \( T \).

2.1. Lemma. Let \( \mathcal{L} \) be a lattice on the compact space \( T \) which is a base for the closed subsets of \( T \). Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( T \). Then there exist disjoint \( A, B \in \mathcal{L} \) such that \( F_1 \subseteq A \) and \( F_2 \subseteq B \).

Proof. For each \( p \in F_2 \) select \( A_p \in \mathcal{L} \) such that \( F_1 \subseteq A_p \) and \( p \in T - A_p \). Then \( \{ T - A_p \mid p \in F_2 \} \) is an open cover for \( F_2 \). Since \( F_2 \) is compact, select \( p_1, \ldots, p_n \in F_2 \) such that \( \{ T - A_{p_k} \mid k = 1, \ldots, n \} \) is a finite open cover for \( F_2 \). Define \( A = \bigcap \{ A_{p_k} \mid k = 1, \ldots, n \} \). Then \( A \in \mathcal{L} \), \( F_1 \subseteq A \) and \( A \cap F_2 = \emptyset \). Applying this procedure to \( F_2 \) and \( A \), we obtain \( B \in \mathcal{L} \) such that \( F_1 \subseteq A \), \( F_2 \subseteq B \) and \( A \cap B = \emptyset \).

2.2. Theorem. Let \( T \) be a compact space. Then \( \mathcal{L} \) is a Wallman base on \( T \) iff \( \mathcal{L} \) is a lattice on \( T \) which is a base for the closed subsets of \( T \).

Proof. Suppose \( \mathcal{L} \) is a lattice on \( T \) which is a base for the closed sets. By 2.1, \( \mathcal{L} \) separates points and closed sets in \( T \). It then suffices to show that \( \mathcal{L} \) is a normal lattice. Let \( A \) and \( B \) be disjoint \( \mathcal{L} \)-sets. Since \( T \) is a normal space, select disjoint open sets \( G \) and \( H \) in \( T \) covering \( A \) and \( B \), respectively. Since \( A \) and \( T - G \) are disjoint closed subsets of \( T \), as are \( B \) and \( T - H \), we use 2.1 to obtain \( C, D \in \mathcal{L} \) such that \( T - G \subseteq C \), \( A \cap C = \emptyset \), \( T - H \subseteq D \) and \( B \cap D = \emptyset \). Then \( A \subseteq T - C \), \( B \subseteq T - D \) and \( C \cup D = T \); i.e., \( \mathcal{L} \) is a normal lattice. This completes the proof.

For \( T \) a compactification of \( X \), we now provide a necessary and sufficient condition that \( T = \omega X \), where \( \mathcal{L} \) is a Wallman base on \( T \) and \( \mathcal{L}_X = \{ A \cap X \mid A \in \mathcal{L} \} \). To facilitate the discussion, we use the notation: for \( T \) a compactification of \( X \), \( \mathcal{L} \) a lattice on \( X \) and \( \mathcal{U} \) an \( \mathcal{L} \)-ultrafilter, \( \mathcal{U} \rightarrow t \) means \( \bigcap \{ \text{cl}_T A \mid A \in \mathcal{U} \} = \{ t \} \).

2.3. Lemma. Let \( T \in cX \) and \( \mathcal{L} \) a lattice on \( X \). Suppose for each \( t \in T \) there exists \( \mathcal{U} \in \omega \mathcal{L} \) such that \( \mathcal{U} \rightarrow t \). Then \( \text{cl}_T A \cap \text{cl}_T B = \emptyset \) whenever \( A, B \in \mathcal{L} \) and \( A \cap B = \emptyset \). Let \( A, B \in \mathcal{L} \). It suffices to show that \( \text{cl}_T A \cap \text{cl}_T B \subseteq \text{cl}_T (A \cap B) \). This is obvious if \( \text{cl}_T A \cap \text{cl}_T B \subseteq \emptyset \). Suppose \( t \in \text{cl}_T A \cap \text{cl}_T B \). Select \( \mathcal{U} \in \omega \mathcal{L} \) such that \( \mathcal{U} \rightarrow t \). We show \( A \in \mathcal{U} \). If not, select \( C \in \mathcal{U} \) such that \( A \cap C = \emptyset \) [2, 2.1]. Then \( \text{cl}_T A \cap \text{cl}_T C = \emptyset \) by hypothesis. But \( t \in \text{cl}_T A \cap \text{cl}_T C \) since \( C \in \mathcal{U} \) and \( \mathcal{U} \rightarrow t \). From this contradiction, we
conclude $A \subseteq \mathcal{U}$. Similarly $B \subseteq \mathcal{U}$. Hence, $A \cap B \subseteq \mathcal{U}$ and $t \in \text{cl}_T(A \cap B)$. This completes the proof since the converse is obvious.

2.4. **Theorem.** Let $T \subseteq \mathcal{X}$ and $\mathcal{L}$ a Wallman base on $T$. Then $T = \omega \mathcal{L}_\lambda$ iff $\text{cl}_T A \cap \text{cl}_T B = \varnothing$ whenever $A, B \in \mathcal{L}_\lambda$ and $A \cap B = \varnothing$.

**Proof.** Suppose $\text{cl}_T A \cap \text{cl}_T B = \varnothing$ whenever $A, B \in \mathcal{L}_\lambda$ and $A \cap B = \varnothing$.

First, we show for each $t \in T$ there exists $\mathcal{U} \in \omega \mathcal{L}_\lambda$ such that $\mathcal{U} \to t$. Let $t \in T$ and define $\mathcal{F} = \{B \in \mathcal{L} \mid t \in \text{int}_T B\}$. Then $\mathcal{F}$ is an $\mathcal{L}$-filter and $t \in \text{int}_T(\text{cl}_T(B \cap X))$ for each $B \in \mathcal{F}$. Since $\mathcal{F}_X = \{B \cap X \mid B \in \mathcal{F}\}$ has the finite intersection property, select $\mathcal{U} \in \mathcal{L}_\lambda$ such that $\mathcal{U} \subseteq \mathcal{F}_X$. Then $\mathcal{U} \not= \cap(\text{cl}_T V \mid V \in \mathcal{U}) \subseteq \cap(\text{cl}_T B \mid B \in \mathcal{F}_X) = \{t\}$. Hence, $\mathcal{U} \in \omega \mathcal{L}_\lambda$ and $\mathcal{U} \to t$. By 2.3, condition (i) of Brooks' theorem [2, Theorem 4.2, p. 167] holds.

Second, we show $\mathcal{L}_\lambda$ is a normal lattice on $X$. Let $A, B \in \mathcal{L}$ where $A \cap B \cap X = \varnothing$. Then $\text{cl}_T(A \cap X) \cap \text{cl}_T(B \cap X) = \varnothing$ by hypothesis. Using 2.1, select $A', B' \in \mathcal{L}$ such that $\text{cl}_T(A \cap X) \subseteq A'$, $\text{cl}_T(B \cap X) \subseteq B'$ and $A' \cap B' = \varnothing$. Since $\mathcal{L}$ is a normal base on $T$, select $C, D \in \mathcal{L}$ such that $A' \subseteq T - C$, $B' \subseteq T - D$ and $C \cup D = T$. Since $\text{cl}_T(A \cap X) \subseteq T - C$ and $\text{cl}_T(B \cap X) \subseteq T - D$, then we have $A \cap X \subseteq X - (C \cap X)$, $B \cap X \subseteq X - (D \cap X)$, $(C \cap X) \cup (D \cap X) = X$ and $C \cap X, D \cap X \in \mathcal{L}_X$. Hence, $\mathcal{L}_X$ is a normal lattice on $X$.

Using the above techniques, conditions (ii) and (iii) of Brooks' theorem [2, Theorem 4.2] are similarly verified. Hence, $T = \omega \mathcal{L}_\lambda$. Since the converse holds immediately from Brooks' theorem, the proof is complete.

3. **Dense Wallman bases.** Let $\mathcal{L}$ be a lattice on $T$. For $S \subseteq T$, define $\mathcal{L}_S = \{A \cap S \mid A \in \mathcal{L}\}$. Also, $B$ is a co-$\mathcal{L}$ set in $T$ means $B$ is the complement of an $\mathcal{L}$-set; i.e., there is an $\mathcal{L}$-set $A$ such that $B = T - A$ (denote $B = \text{co}_T A$).

3.1. **Theorem.** Let $\mathcal{L}$ be a Wallman base on the space $T$. Then $\mathcal{L}_X$ is a Wallman base for each subspace $X$ of $T$ iff $\mathcal{L}_Y$ is a Wallman base for each co-$\mathcal{L}$ set $Y$ in $T$.

**Proof.** Suppose $\mathcal{L}_Y$ is a Wallman base on $Y$ for each co-$\mathcal{L}$ set $Y$ in $T$. Let $X$ be a subspace of $T$. Then $\mathcal{L}_X$ is a lattice on $X$ which is a base for the closed subsets of $X$. It remains to show that $\mathcal{L}_X$ is normal. Let $A, B \in \mathcal{L}_X$ such that $A \cap B = \varnothing$. Select $A_1, B_1 \in \mathcal{L}$ such that $A = A_1 \cap X$ and $B = B_1 \cap X$. Then $X \subseteq S = \text{co}_T(A_1 \cap B_1)$. Let $A_2 = A_1 \cap S$ and $B_2 = B_1 \cap S$. Then $A_2 \cap B_2 = \varnothing$. Since $\mathcal{L}_S$ is a Wallman base on $S$, select $C_1, D_1 \in \mathcal{L}$ such that $C = C_1 \cap S$, $D = D_1 \cap S \subseteq \mathcal{L}_S$ where $A_2 \subseteq \text{co}_S C$, $B_2 \subseteq \text{co}_S D$ and $C \cap D = \varnothing$. Hence, $\mathcal{L}_X$ is normal.
$B_2 \subseteq \cos D$ and $C \cup D = S$. Now $\cos E \cap \cos D \cap S = \emptyset$ since $(\cos E) \cap S = \cos (E \cap S)$ for each $E \subseteq \mathcal{E}$. Thus $A \subseteq \cos (C \cap X)$, $B \subseteq \cos (D \cap X)$ and $(C \cap X) \cup (D \cap X) = X$. Hence, $\mathcal{E}_X$ is a normal base on $X$.

Similarly we obtain

3.2. **Theorem.** Let $\mathcal{E}$ be a Wallman base on the space $T$. Then $\mathcal{E}_X$ is a Wallman base for each dense subspace $X$ of $T$ if $\mathcal{E}_Y$ is a Wallman base for each dense co-$\mathcal{E}$ set $Y$ in $T$.

3.3. **Theorem.** Let $T$ be a compact space and $\mathcal{E}$ a Wallman base on $T$. Then $T = w\mathcal{E}_X$ for each dense subspace $X$ of $T$ if $T = w\mathcal{E}_Y$ for each dense co-$\mathcal{E}$ set $Y$ in $T$.

**Proof.** Suppose $T = w\mathcal{E}_Y$ for each dense co-$\mathcal{E}$ set $Y$ in $T$ and let $X$ be a dense subspace of $T$. We apply 2.4. Let $A, B \subseteq \mathcal{E}_X$ such that $A \cap B = \emptyset$. Select $A_1, B_1 \subseteq \mathcal{E}$ such that $A = A_1 \cap X$ and $B = B_1 \cap X$. Then $Y = \cos (A_1 \cap B_1)$ is a dense co-$\mathcal{E}$ set in $T$. Define $A_2 = A_1 \cap Y$ and $B_2 = B_1 \cap Y$. Since $A_2 \cap B_2 = \emptyset$ and $T = w\mathcal{E}_Y$, then $\cl_T A_2 \cap \cl_T B_2 = \emptyset$ by 2.4. But $A \subseteq \cl_T A_2$ and $B \subseteq \cl_T B_2$; so, $\cl_T A \cap \cl_T B = \emptyset$. Hence, $T = w\mathcal{E}_X$ by 2.4.

4. **Compact $F$-spaces.** We now apply the previous theory to show that every compact $F$-space (i.e., disjoint cozero sets are completely separated [GJ]) is a $\varepsilon$-compactification of each of its dense subspaces. The ring of all continuously extendable functions from a space $X$ to an extension $T$ is denoted by $E(X, T)$.

4.1. **Theorem.** Let $T$ be a compact $F$-space. Then $T$ is a $\varepsilon$-compactification of each of its dense subspaces.

**Proof.** We apply 3.3. Now $Z[T]$ is a Wallman base on $T$. Let $Y$ be a dense cozero set of $T$. Then $C^*(Y) = E(Y, T)$ [GJ, 14.25 (5)] and so $T = \cl_T Y = \beta Y$ [GJ, 6.5 (11)]. Also $\beta Y = wZ(Y)$ [GJ, 6.5], so $T = wZ(Y)$. And $Z(Y) = \{Z \cap Y \mid Z \subseteq Z(T)\}$ since $Y$ is $\varepsilon$-embedded in $T$ [5, Theorem 3]. By 3.3, $T = wZ(X)$ for each dense subspace $X$ in $T$. This completes the proof.

We remark that if $X$ is $\sigma$-compact and locally compact, then $\beta X - X$ is a compact $F$-space [GJ, 14.27]. So $\beta R - R$ and $\beta N - N$ are z-compactifications of each of their dense subspaces, where $R$ denotes the reals and $N$ the positive integers with their usual topologies. Also if $Y$ is a dense cozero set of a compact space $T$, then $\beta Y - Y$ is a $\varepsilon$-compactification of each of its dense subspaces [GJ, 14(0.2)].

5. **Orderable compact spaces.**

5.1. **Theorem.** Let $T$ be an ordered compact space. Then $T$ is a Wall-
man compactification of each of its dense subspaces.

Proof. Let \( \leq \) be an order on \( T \) compatible with the topology for \( T \). Let \( X \) be a dense subspace of \( T \). The left endpoint \( a \) of the interval \( [a, b] \) in \( T \) is admissible means either \( a \in X \) or \( [a, b] \) is a neighborhood of \( a \) in \( T \). The right endpoint \( b \) of the interval \( [a, b] \) in \( T \) is admissible means either \( b \in X \) or \( [a, b] \) is a neighborhood of \( b \) in \( T \). Let \( \mathcal{L} \) be the lattice on \( T \) of finite unions of closed intervals with admissible endpoints. Clearly \( X \) is \( \mathcal{L} \)-dense. By 2.2 and [1, Theorems 3] it then suffices to show that \( \mathcal{L} \) is a base for the closed sets in \( T \). Let \( F \) be a closed subset of \( T \) and \( p \in T - F \).

Case 1. \( p < t \) for each \( t \in F \). Let \( t' = \inf_T F \). If \( t' \in X \), then \( [t', 1_T] \in \mathcal{L} \) (\( 1_T = \sup_T T \)), \( F \subset [t', 1_T] \) and \( p \notin [t', 1_T] \). Suppose \( t' \notin X \). If \( x \in (p, t') \cap X \), then \( [x, 1_T] \in \mathcal{L} \), \( F \subset [x, 1_T] \) and \( p \notin [x, 1_T] \). If \( (p, t') \cap X = \emptyset \), then \( [t', 1_T] \) is a \( T \)-neighborhood of \( t' \). So \( [t', 1_T] \in \mathcal{L} \), \( F \subset [t', 1_T] \) and \( p \notin [t', 1_T] \).

Case 2. \( t < p \) for each \( t \in F \). Let \( t' = \sup_T F \) and proceed as in Case 1 obtaining \( q \in [t', p] \) such that \( [0_T, q] \in \mathcal{L} \) (\( 0_T = \inf_T T \)), \( F \subset [0_T, q] \) and \( p \notin [0_T, q] \).

Case 3. There exists \( q_1, q_2 \in F \) such that \( q_1 < p < q_2 \). Let \( F_1 = \{ t \in F \mid p < t \} \) and \( F_2 = \{ t \in F \mid t < p \} \). Then \( F_1, F_2 \) are disjoint closed subsets of \( T \) and \( F_1 \cup F_2 = F \). Apply Case 1 to \( p \) and \( F_1 \) obtaining \( t_1 \in T \) such that \( [t_1, 1_T] \in \mathcal{L} \), \( F_1 \subset [t_1, 1_T] \) and \( p \notin [t_1, 1_T] \). Apply Case 2 to \( F_2 \) and \( p \) obtaining \( t_2 \in T \) such that \( [0_T, t_2] \in \mathcal{L} \), \( F_2 \subset [0_T, t_2] \) and \( p \notin [0_T, t_2] \). Then \( [0_T, t_2] \cup [t_1, 1_T] \in \mathcal{L} \), \( F \subset [0_T, t_2] \cup [t_1, 1_T] \) and \( p \notin [0_T, t_2] \cup [t_1, 1_T] \). Hence, \( \mathcal{L} \) is a base for the closed subsets of \( T \). This completes the proof.

5.2. Theorem. Let \( X \) be an ordered space and let \( T \) be the maximal ordered compactification of \( X \). Then \( T \in \varepsilon X \).

Proof. For the existence of \( T \), see [6]. \( T \) has the following property: if \( t \in T - X \), then there exists \( Y \subset X \) such that either \( t = \sup_T Y \) or \( t = \inf_T Y \), but never both. Let \( \mathcal{L} \) be the lattice on \( T \) of finite unions of sets of the form \( f^{-1}(a) \) where \( f : T \to [0, 1] \) is continuous and increasing, \( a \in [0, 1] \) and if \( t \in f^{-1}(a) \cap (T - X) \), then \( f^{-1}(a) \) is a neighborhood of \( t \) in \( T \). Using the method of the previous theorem, it is straightforward to show that \( \mathcal{L} \) is a Wallman base on \( T \) and that \( X \) is \( \mathcal{L} \)-dense. By [1, Theorem 3], \( T = w\mathcal{L}X \). Hence, \( T \in \varepsilon X \) since \( \mathcal{L} \subset Z(T) \).

Although we have shown that the maximal ordered compactification of an ordered space is a \( \varepsilon \)-compactification, we have not been able to show this for arbitrary ordered compactifications.

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