WALLMAN-TYPE COMPACTIFICATIONS

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ABSTRACT. All spaces in this paper are Tychonoff. A Wallman base on a space X is a normal separating ring of closed subsets of X (see Steiner, Duke Math. J. **35** (1968), 269–276). Let T be a compact space and $\mathcal{L}$ a Wallman base on T. For $X \subset T$, define $\mathcal{L}_X = \{A \cap X \mid A \in \mathcal{L}\}$.

THEOREM 1. *If X is a dense subspace of T, then $T = w\mathcal{L}_X$ iff $\mathrm{cl}_T A \cap \mathrm{cl}_T B = \emptyset$ whenever $A, B \in \mathcal{L}_X$ and $A \cap B = \emptyset$.*

THEOREM 2. *$T = w\mathcal{L}_X$ for each dense $X \subset T$ iff $T = w\mathcal{L}_Y$ for each dense $Y \subset T$ where $T - Y \in \mathcal{L}$.*

From these theorems we show that every compact F-space and every compact orderable space is a Wallman compactification of each of its dense subspaces.

1. **Introduction.** The Wallman base concept, first initiated by H. Wallman [**W**] in 1938, has been successfully used by Frink [**3**], Alo and Shapiro [**1**], Brooks [**2**], Hager [**4**], E. Steiner and A. Steiner [**8**], [**9**], and others. An important reference is the text *Rings of continuous functions* [**GJ**] which exploits Wallman's method in constructing the Stone-Čech compactification. We shall rely on these references for background, definitions and terminology.

The chief motivation for this paper is an answer to the question: Can every Hausdorff compactification of a Tychonoff space be constructed by Wallman's method? We provide some partial solutions.

All topological spaces in this paper are Tychonoff. The space T is a compactification of X (denoted $T \in cX$) means T is a compact extension of X. When referring to an extension T of X, we may (and do) take X as a subspace of T (tacitly assuming the necessary embedding maps). Accordingly, we shall write $T_1 = T_2$ when T_1 and T_2 are equivalent compactifications of a given space. The notation $T \in wX$ ($T \in zX$) means T is a Wallman compactification (z-compactification) of X. We reserve βX for the Stone-Čech compactification of X. Finally, $\mathrm{cl}_X A$ and $\mathrm{int}_X A$ denote the closure and interior, respectively, of A in X.

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2. **Wallman bases on compact spaces.** The collection $\mathcal{L}$ of subsets of X is a lattice on X means that $\emptyset, X\in\mathcal{L}$ and $\mathcal{L}$ is closed under finite intersections and unions [**2**]. We first provide necessary and sufficient conditions for a lattice on a compact space T to be a Wallman base on T.

2.1. LEMMA. *Let $\mathcal{L}$ be a lattice on the compact space T which is a base for the closed subsets of T. Let F_1 and F_2 be disjoint closed subsets of T. Then there exist disjoint A, $B\in\mathcal{L}$ such that $F_1\subset A$ and $F_2\subset B$.*

PROOF. For each $p\in F_2$ select $A_p\in\mathcal{L}$ such that $F_1\subset A_p$ and $p\in T-A_p$. Then $\{T-A_p \mid p\in F_2\}$ is an open cover for F_2. Since F_2 is compact, select $p_1, \cdots, p_n\in F_2$ such that $\{T-A_{p_k} \mid k=1, \cdots, n\}$ is a finite open cover for F_2. Define $A=\bigcap\{A_{p_k} \mid k=1, \cdots, n\}$. Then $A\in\mathcal{L}$, $F_1\subset A$ and $A\cap F_2=\emptyset$. Applying this procedure to F_2 and A, we obtain $B\in\mathcal{L}$ such that $F_1\subset A$, $F_2\subset B$ and $A\cap B=\emptyset$.

2.2. THEOREM. *Let T be a compact space. Then $\mathcal{L}$ is a Wallman base on T iff $\mathcal{L}$ is a lattice on T which is a base for the closed subsets of T.*

PROOF. Suppose $\mathcal{L}$ is a lattice on T which is a base for the closed sets. By 2.1, $\mathcal{L}$ separates points and closed sets in T. It then suffices to show that $\mathcal{L}$ is a normal lattice. Let A and B be disjoint $\mathcal{L}$-sets. Since T is a normal space, select disjoint open sets G and H in T covering A and B, respectively. Since A and $T-G$ are disjoint closed subsets of T, as are B and $T-H$, we use 2.1 to obtain C, $D\in\mathcal{L}$ such that $T-G\subset C$, $A\cap C=\emptyset$, $T-H\subset D$ and $B\cap D=\emptyset$. Then $A\subset T-C$, $B\subset T-D$ and $C\cup D=T$; i.e., $\mathcal{L}$ is a normal lattice. This completes the proof.

For T a compactification of X, we now provide a necessary and sufficient condition that $T=w\mathcal{L}_X$, where $\mathcal{L}$ is a Wallman base on T and $\mathcal{L}_X=\{A\cap X \mid A\in\mathcal{L}\}$. To facilitate the discussion, we use the notation: for T a compactification of X, $\mathcal{L}$ a lattice on X and $\mathcal{U}$ an $\mathcal{L}$-ultrafilter, $\mathcal{U}\to t$ means $\bigcap\{\mathrm{cl}_T A \mid A\in\mathcal{U}\}=\{t\}$.

2.3. LEMMA. *Let $T\in cX$ and $\mathcal{L}$ a lattice on X. Suppose for each $t\in T$ there exists $\mathcal{U}\in w\mathcal{L}$ such that $\mathcal{U}\to t$. Then $\mathrm{cl}_T A\cap\mathrm{cl}_T B=\emptyset$ whenever A, $B\in\mathcal{L}$ and $A\cap B=\emptyset$ iff $\mathrm{cl}_T(A\cap B)=\mathrm{cl}_T A\cap\mathrm{cl}_T B$ for each A, $B\in\mathcal{L}$.*

PROOF. Suppose $\mathrm{cl}_T A\cap\mathrm{cl}_T B=\emptyset$ whenever A, $B\in\mathcal{L}$ and $A\cap B=\emptyset$. Let A, $B\in\mathcal{L}$. It suffices to show that $\mathrm{cl}_T A\cap\mathrm{cl}_T B\subset\mathrm{cl}_T(A\cap B)$. This is obvious if $\mathrm{cl}_T A\cap\mathrm{cl}_T B=\emptyset$. Suppose $t\in\mathrm{cl}_T A\cap\mathrm{cl}_T B$. Select $\mathcal{U}\in w\mathcal{L}$ such that $\mathcal{U}\to t$. We show $A\in\mathcal{U}$. If not, select $C\in\mathcal{U}$ such that $A\cap C=\emptyset$ [**2**, 2.1]. Then $\mathrm{cl}_T A\cap\mathrm{cl}_T C=\emptyset$ by hypothesis. But $t\in\mathrm{cl}_T A\cap\mathrm{cl}_T C$ since $C\in\mathcal{U}$ and $\mathcal{U}\to t$. From this contradiction, we conclude $A\in\mathfrak{U}$. Similarly $B\in\mathfrak{U}$. Hence, $A\cap B\in\mathfrak{U}$ and $t\in\mathrm{cl}_T(A\cap B)$. This completes the proof since the converse is obvious.

2.4. THEOREM. *Let $T\in cX$ and $\mathfrak{L}$ a Wallman base on T. Then $T=w\mathfrak{L}_X$ iff* $\mathrm{cl}_TA\cap\mathrm{cl}_TB=\varnothing$ *whenever A, $B\in L_X$* and $A\cap B=\varnothing$.

PROOF. Suppose $\mathrm{cl}_TA\cap\mathrm{cl}_TB=\varnothing$ whenever A, $B\in\mathfrak{L}_X$ and $A\cap B=\varnothing$.

First, we show for each $t\in T$ there exists $\mathfrak{U}\in w\mathfrak{L}_X$ such that $\mathfrak{U}\to t$. Let $t\in T$ and define $\mathfrak{F}=\{B\in\mathfrak{L}\mid t\in\mathrm{int}_TB\}$. Then $\mathfrak{F}$ is an $\mathfrak{L}$-filter and $t\in\mathrm{int}_T(\mathrm{cl}_T(B\cap X))$ for each $B\in\mathfrak{F}$. Since $\mathfrak{F}_X=\{B\cap X\mid B\in\mathfrak{F}\}$ has the finite intersection property, select $\mathfrak{U}\in\mathfrak{L}_X$ such that $\mathfrak{F}_X\subset\mathfrak{U}$. Then $\varnothing\neq\bigcap(\mathrm{cl}_TV\mid V\in\mathfrak{U}\}\subset\bigcap\{\mathrm{cl}_TB\mid B\in\mathfrak{F}_X\}=\{t\}$. Hence, $\mathfrak{U}\in w\mathfrak{L}_X$ and $\mathfrak{U}\to t$. By 2.3, condition (i) of Brooks' theorem [2, Theorem **4**.2, p. 167] holds.

Second, we show $\mathfrak{L}_X$ is a normal lattice on X. Let A, $B\in\mathfrak{L}$ where $A\cap B\cap X=\varnothing$. Then $\mathrm{cl}_T(A\cap X)\cap\mathrm{cl}_T(B\cap X)=\varnothing$ by hypothesis. Using 2.1, select A', $B'\in\mathfrak{L}$ such that $\mathrm{cl}_T(A\cap X)\subset A'$, $\mathrm{cl}_T(B\cap X)\subset B'$ and $A'\cap B'=\varnothing$. Since $\mathfrak{L}$ is a normal base on T, select C, $D\in\mathfrak{L}$ such that $A'\subset T-C$, $B'\subset T-D$ and $C\cup D=T$. Since $\mathrm{cl}_T(A\cap X)\subset T-C$ and $\mathrm{cl}_T(B\cap X)\subset T-D$, then we have $A\cap X\subset X-(C\cap X)$, $B\cap X\subset X-(D\cap X)$, $(C\cap X)\cup(D\cap X)=X$ and $C\cap X$, $D\cap X\in\mathfrak{L}_X$. Hence, $\mathfrak{L}_X$ is a normal lattice on X.

Using the above techniques, conditions (ii) and (iii) of Brooks' theorem [**2**, Theorem 4.2] are similarly verified. Hence, $T=w\mathfrak{L}_X$. Since the converse holds immediately from Brooks' theorem, the proof is complete.

3. **Dense Wallman bases.** Let $\mathfrak{L}$ be a lattice on T. For $S\subset T$, define $\mathfrak{L}_S=\{A\cap S\mid A\in\mathfrak{L}\}$. Also, B is a co-$\mathfrak{L}$ set in T means B is the complement of an $\mathfrak{L}$-set; i.e., there is an $\mathfrak{L}$-set A such that $B=T-A$ (denote $B=\mathrm{co}_TA$).

3.1. THEOREM. *Let $\mathfrak{L}$ be a Wallman base on the space T. Then $\mathfrak{L}_X$ is a Wallman base for each subspace X of T iff $\mathfrak{L}_Y$ is a Wallman base for each co-$\mathfrak{L}$ set Y in T.*

PROOF. Suppose $\mathfrak{L}_Y$ is a Wallman base on Y for each co-$\mathfrak{L}$ set Y in T. Let X be a subspace of T. Then $\mathfrak{L}_X$ is a lattice on X which is a base for the closed subsets of X. It remains to show that $\mathfrak{L}_X$ is normal. Let A, $B\in\mathfrak{L}_X$ such that $A\cap B=\varnothing$. Select A_1, $B_1\in\mathfrak{L}$ such that $A=A_1\cap X$ and $B=B_1\cap X$. Then $X\subset S=\mathrm{co}_T(A_1\cap B_1)$. Let $A_2=A_1\cap S$ and $B_2=B_1\cap S$. Then $A_2\cap B_2=\varnothing$. Since $\mathfrak{L}_S$ is a Wallman base on S, select C_1, $D_1\in\mathfrak{L}$ such that $C=C_1\cap S$, $D=D_1\cap S\in\mathfrak{L}_S$ where $A_2\subset\mathrm{co}_SC$,

$B_2 \subset \mathrm{co}_S D$ and $C \cup D = S$. Now $\mathrm{co}_T C_1 \cap \mathrm{co}_T D_1 \cap S = \emptyset$ since $(\mathrm{co}_T E) \cap S = \mathrm{co}_T(E \cap S)$ for each $E \in \mathfrak{L}$. Thus $A \subset \mathrm{co}_X(C_1 \cap X)$, $B \subset \mathrm{co}_X(D_1 \cap X)$ and $(C_1 \cap X) \cup (D_1 \cap X) = X$. Hence, $\mathfrak{L}_X$ is a normal base on X.

Similarly we obtain

3.2. THEOREM. *Let $\mathfrak{L}$ be a Wallman base on the space T. Then $\mathfrak{L}_X$ is a Wallman base for each dense subspace X of T iff $\mathfrak{L}_Y$ is a Wallman base for each dense co-$\mathfrak{L}$ set Y in T.*

3.3. THEOREM. *Let T be a compact space and $\mathfrak{L}$ a Wallman base on T. Then $T = w\mathfrak{L}_X$ for each dense subspace X of T iff $T = w\mathfrak{L}_Y$ for each dense co-$\mathfrak{L}$ set Y in T.*

PROOF. Suppose $T = w\mathfrak{L}_Y$ for each dense co-$\mathfrak{L}$ set Y in T and let X be a dense subspace of T. We apply 2.4. Let $A, B \in \mathfrak{L}_X$ such that $A \cap B = \emptyset$. Select $A_1, B_1 \in \mathfrak{L}$ such that $A = A_1 \cap X$ and $B = B_1 \cap X$. Then $Y = \mathrm{co}_T(A_1 \cap B_1)$ is a dense co-$\mathfrak{L}$ set in T. Define $A_2 = A_1 \cap Y$ and $B_2 = B_1 \cap Y$. Since $A_2 \cap B_2 = \emptyset$ and $T = w\mathfrak{L}_Y$, then $\mathrm{cl}_T A_2 \cap \mathrm{cl}_T B_2 = \emptyset$ by 2.4. But $A \subset \mathrm{cl}_T A_2$ and $B \subset \mathrm{cl}_T B_2$; so, $\mathrm{cl}_T A \cap \mathrm{cl}_T B = \emptyset$. Hence, $T = w\mathfrak{L}_X$ by 2.4.

4. **Compact F-spaces.** We now apply the previous theory to show that every compact F-space (i.e., disjoint cozero sets are completely separated [**GJ**]) is a z-compactification of each of its dense subspaces. The ring of all continuously extendable functions from a space X to an extension T is denoted by $E(X, T)$.

4.1. THEOREM. *Let T be a compact F-space. Then T is a z-compactification of each of its dense subspaces.*

PROOF. We apply 3.3. Now $Z[T]$ is a Wallman base on T. Let Y be a dense cozero set of T. Then $C^*(Y) = E(Y, T)$ [**GJ**, 14.25 (5)] and so $T = \mathrm{cl}_T Y = \beta Y$ [**GJ**, 6.5 (II)]. Also $\beta Y = wZ(Y)$ [**GJ**, 6.5], so $T = wZ(Y)$. And $Z(Y) = \{Z \cap Y \mid Z \in Z(T)\}$ since Y is z-embedded in T [**5**, Theorem 3]. By 3.3, $T = wZ(X)$ for each dense subspace X in T. This completes the proof.

We remark that if X is σ-compact and locally compact, then $\beta X - X$ is a compact F-space [**GJ**, 14.27]. So $\beta \boldsymbol{R} - \boldsymbol{R}$ and $\beta \boldsymbol{N} - \boldsymbol{N}$ are z-compactifications of each of their dense subspaces, where $\boldsymbol{R}$ denotes the reals and $\boldsymbol{N}$ the positive integers with their usual topologies. Also if Y is a dense cozero set of a compact space T, then $\beta Y - Y$ is a z-compactification of each of its dense subspaces [**GJ**, 14(0.2)].

5. **Orderable compact spaces.**

5.1. THEOREM. *Let T be an ordered compact space. Then T is a Wall-*

man compactification of each of its dense subspaces.

Proof. Let $\leqq$ be an order on T compatible with the topology for T. Let X be a dense subspace of T. The left endpoint a of the interval $[a, b]$ in T is admissable means either $a \in X$ or $[a, b]$ is a neighborhood of a in T. The right endpoint b of the interval $[a, b]$ in T is admissable means either $b \in X$ or $[a, b]$ is a neighborhood of b in T. Let $\mathfrak{L}$ be the lattice on T of finite unions of closed intervals with admissable endpoints. Clearly X is $\mathfrak{L}$-dense. By 2.2 and [**1**, Theorems 3] it then suffices to show that $\mathfrak{L}$ is a base for the closed sets in T. Let F be a closed subset of T and $p \in T - F$.

Case 1. $p < t$ for each $t \in F$. Let $t' = \inf_T F$. If $t' \in X$, then $[t', 1_T] \in \mathfrak{L}$ $(1_T = \sup_T T)$, $F \subset [t', 1_T]$ and $p \notin [t', 1_T]$. Suppose $t' \notin X$. If $x \in (p, t') \cap X$, then $[x, 1_T] \in \mathfrak{L}$, $F \subset [x, 1_T]$ and $p \notin [x, 1_T]$. If $(p, t') \cap X = \emptyset$, then $[t', 1_T]$ is a T-neighborhood of t'. So $[t', 1_T] \in \mathfrak{L}$, $F \subset [t', 1_T]$ and $p \notin [t', 1_T]$.

Case 2. $t < p$ for each $t \in F$. Let $t' = \sup_T F$ and proceed as in Case 1 obtaining $q \in [t', p)$ such that $[0_T, q] \in \mathfrak{L}$ $(0_T = \inf_T T)$, $F \subset [0_T, q]$ and $p \notin [0_T, q]$.

Case 3. There exists $q_1, q_2 \in F$ such that $q_1 < p < q_2$. Let $F_1 = \{t \in F \mid p < t\}$ and $F_2 = \{t \in F \mid t < p\}$. Then F_1, F_2 are disjoint closed subsets of T and $F_1 \cup F_2 = F$. Apply Case 1 to p and F_1 obtaining $t_1 \in T$ such that $[t_1, 1_T] \in \mathfrak{L}$, $F_1 \subset [t_1, 1_T]$ and $p \notin [t_1, 1_T]$. Apply Case 2 to F_2 and p obtaining $t_2 \in T$ such that $[0_T, t_2] \in \mathfrak{L}$, $F_2 \subset [0_T, t_2]$ and $p \notin [0_T, t_2]$. Then $[0_T, t_2] \cup [t_1, 1_T] \in \mathfrak{L}$, $F \subset [0_T, t_2] \cup [t_1, 1_T]$ and $p \notin [0_T, t_2] \cup [t_1, 1_T]$. Hence, $\mathfrak{L}$ is a base for the closed subsets of T. This completes the proof.

5.2. Theorem. *Let X be an ordered space and let T be the maximal ordered compactification of X. Then $T \in zX$.*

Proof. For the existence of T, see [**6**]. T has the following property: if $t \in T - X$, then there exists $Y \subset X$ such that either $t = \sup_T Y$ or $t = \inf_T Y$, but never both. Let $\mathfrak{L}$ be the lattice on T of finite unions of sets of the form $f^{-1}(a)$ where $f: T \to [0, 1]$ is continuous and increasing, $a \in [0, 1]$ and if $t \in f^{-1}(a) \cap (T - X)$, then $f^{-1}(a)$ is a neighborhood of t in T. Using the method of the previous theorem, it is straightforward to show that $\mathfrak{L}$ is a Wallman base on T and that X is $\mathfrak{L}$-dense. By [**1**, Theorem 3], $T = w\mathfrak{L}_X$. Hence, $T \in zX$ since $\mathfrak{L} \subset Z(T)$.

Although we have shown that the maximal ordered compactification of an ordered space is a z-compactification, we have not been able to show this for arbitrary ordered compactifications.

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