WALLMAN-TYPE COMPACTIFICATIONS

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Abstract. All spaces in this paper are Tychonoff. A Wallman base on a space $X$ is a normal separating ring of closed subsets of $X$ (see Steiner, Duke Math. J. 35 (1968), 269–276). Let $T$ be a compact space and $\mathcal{L}$ a Wallman base on $T$. For $X \subset T$, define $\mathcal{L}_X = \{ A \cap X \mid A \in \mathcal{L} \}$.

Theorem 1. If $X$ is a dense subspace of $T$, then $T = w\mathcal{L}_X$ iff $\text{cl}_T A \cap \text{cl}_T B = \emptyset$ whenever $A, B \in \mathcal{L}_X$ and $A \cap B = \emptyset$.

Theorem 2. $T = w\mathcal{L}_X$ for each dense $X \subset T$ iff $T = w\mathcal{L}_Y$ for each dense $Y \subset T$ where $T - Y \in \mathcal{L}$.

From these theorems we show that every compact F-space and every compact orderable space is a Wallman compactification of each of its dense subspaces.

1. Introduction. The Wallman base concept, first initiated by H. Wallman [W] in 1938, has been successfully used by Frink [3], Alo and Shapiro [1], Brooks [2], Hager [4], E. Steiner and A. Steiner [8], [9], and others. An important reference is the text *Rings of continuous functions* [GJ] which exploits Wallman’s method in constructing the Stone-Čech compactification. We shall rely on these references for background, definitions and terminology.

The chief motivation for this paper is an answer to the question: Can every Hausdorff compactification of a Tychonoff space be constructed by Wallman’s method? We provide some partial solutions.

All topological spaces in this paper are Tychonoff. The space $T$ is a compactification of $X$ (denoted $T \subset \bar{X}$) means $T$ is a compact extension of $X$. When referring to an extension $T$ of $X$, we may (and do) take $X$ as a subspace of $T$ (tacitly assuming the necessary embedding maps). Accordingly, we shall write $T_1 = T_2$ when $T_1$ and $T_2$ are equivalent compactifications of a given space. The notation $T \subset wX$ ($T \subset zX$) means $T$ is a Wallman compactification ($z$-compactification) of $X$.

We reserve $\beta X$ for the Stone-Čech compactification of $X$. Finally, $\text{cl}_X A$ and $\text{int}_X A$ denote the closure and interior, respectively, of $A$ in $X$.

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2. Wallman bases on compact spaces. The collection $\mathcal{L}$ of subsets of $X$ is a lattice on $X$ means that $\emptyset, X \in \mathcal{L}$ and $\mathcal{L}$ is closed under finite intersections and unions [2]. We first provide necessary and sufficient conditions for a lattice on a compact space $T$ to be a Wallman base on $T$.

2.1. Lemma. Let $\mathcal{L}$ be a lattice on the compact space $T$ which is a base for the closed subsets of $T$. Let $F_1$ and $F_2$ be disjoint closed subsets of $T$. Then there exist disjoint $A, B \in \mathcal{L}$ such that $F_1 \subseteq A$ and $F_2 \subseteq B$.

Proof. For each $p \in F_2$ select $A_p \in \mathcal{L}$ such that $F_1 \subseteq A_p$ and $p \in T - A_p$. Then $\{ T - A_p | p \in F_2 \}$ is an open cover for $F_2$. Since $F_2$ is compact, select $p_1, \ldots, p_n \in F_2$ such that $\{ T - A_{p_k} | k = 1, \ldots, n \}$ is a finite open cover for $F_2$. Define $A = \bigcap \{ A_{p_k} | k = 1, \ldots, n \}$. Then $A \in \mathcal{L}$, $F_1 \subseteq A$ and $A \cap F_2 = \emptyset$. Applying this procedure to $F_2$ and $A$, we obtain $B \in \mathcal{L}$ such that $F_1 \subseteq B$, $F_2 \subseteq B$ and $B \cap A = \emptyset$.

2.2. Theorem. Let $T$ be a compact space. Then $\mathcal{L}$ is a Wallman base on $T$ iff $\mathcal{L}$ is a lattice on $T$ which is a base for the closed subsets of $T$.

Proof. Suppose $\mathcal{L}$ is a lattice on $T$ which is a base for the closed sets. By 2.1, $\mathcal{L}$ separates points and closed sets in $T$. It then suffices to show that $\mathcal{L}$ is a normal lattice. Let $A$ and $B$ be disjoint $\mathcal{L}$-sets. Since $T$ is a normal space, select disjoint open sets $G$ and $H$ in $T$ covering $A$ and $B$, respectively. Since $A$ and $T - G$ are disjoint closed subsets of $T$, as are $B$ and $T - H$, we use 2.1 to obtain $C, D \in \mathcal{L}$ such that $T - G \subseteq C$, $A \cap C = \emptyset$, $T - H \subseteq D$ and $B \cap D = \emptyset$. Then $A \subseteq T - C$, $B \subseteq T - D$ and $C \cup D = T$; i.e., $\mathcal{L}$ is a normal lattice. This completes the proof.

For $T$ a compactification of $X$, we now provide a necessary and sufficient condition that $T = w\mathcal{L}_X$, where $\mathcal{L}$ is a Wallman base on $T$ and $\mathcal{L}_X = \{ A \cap X | A \in \mathcal{L} \}$. To facilitate the discussion, we use the notation: for $T$ a compactification of $X$, $\mathcal{L}$ a lattice on $X$ and $\mathcal{U}$ an $\mathcal{L}$-ultrafilter, $\mathcal{U} \rightarrow t$ means $\bigcap \{ \text{cl}_T A | A \in \mathcal{U} \} = \{ t \}$.

2.3. Lemma. Let $T \subset cX$ and $\mathcal{L}$ a lattice on $X$. Suppose for each $t \in T$ there exists $\mathcal{U} \subset \mathcal{W}_\mathcal{L}$ such that $\mathcal{U} \rightarrow t$. Then $\text{cl}_T A \cap \text{cl}_T B = \emptyset$ whenever $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$ iff $\text{cl}_T (A \cap B) = \text{cl}_T A \cap \text{cl}_T B$ for each $A, B \in \mathcal{L}$.

Proof. Suppose $\text{cl}_T A \cap \text{cl}_T B = \emptyset$ whenever $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$. Let $A, B \in \mathcal{L}$. It suffices to show that $\text{cl}_T A \cap \text{cl}_T B \subseteq \text{cl}_T (A \cap B)$. This is obvious if $\text{cl}_T A \cap \text{cl}_T B = \emptyset$. Suppose $t \in \text{cl}_T A \cap \text{cl}_T B$. Select $\mathcal{U} \subset \mathcal{W}_\mathcal{L}$ such that $\mathcal{U} \rightarrow t$. We show $A \in \mathcal{U}$. If not, select $C \in \mathcal{U}$ such that $A \cap C = \emptyset$ [2, 2.1]. Then $\text{cl}_T A \cap \text{cl}_T C = \emptyset$ by hypothesis. But $t \in \text{cl}_T A \cap \text{cl}_T C$ since $C \in \mathcal{U}$ and $\mathcal{U} \rightarrow t$. From this contradiction, we
conclude \( A \in \mathcal{U} \). Similarly \( B \in \mathcal{U} \). Hence, \( A \cap B \in \mathcal{U} \) and \( t \in \text{cl}_T(A \cap B) \). This completes the proof since the converse is obvious.

2.4. Theorem. Let \( T \in \mathcal{L}_X \) and \( \mathcal{L} \) a Wallman base on \( T \). Then \( T = \omega \mathcal{L}_X \) iff \( \text{cl}_T A \cap \text{cl}_T B = \emptyset \) whenever \( A, B \in \mathcal{L}_X \) and \( A \cap B = \emptyset \).

Proof. Suppose \( \text{cl}_T A \cap \text{cl}_T B = \emptyset \) whenever \( A, B \in \mathcal{L}_X \) and \( A \cap B = \emptyset \).

First, we show for each \( t \in T \) there exists \( \mathcal{U} \in \omega \mathcal{L}_X \) such that \( \mathcal{U} \rightarrow t \).

Let \( t \in T \) and define \( \mathcal{F}_t = \{ B \in \mathcal{L} \mid t \in \text{int}_T B \} \). Then \( \mathcal{F}_t \) is an \( \mathcal{L} \)-filter and \( t \in \text{int}_T(\text{cl}_T(B \cap X)) \) for each \( B \in \mathcal{F}_t \). Since \( \mathcal{F}_X = \{ B \cap X \mid B \in \mathcal{F}_t \} \) has the finite intersection property, select \( \mathcal{U} \in \mathcal{L}_X \) such that \( \mathcal{U} \subset \mathcal{F}_X \). Then \( \mathcal{F}_X = \{ \text{cl}_T B \mid B \in \mathcal{F}_X \} = \{ t \} \). Hence, \( \mathcal{U} \in \omega \mathcal{L}_X \) and \( \mathcal{U} \rightarrow t \). By 2.3, condition (i) of Brooks' theorem [2, Theorem 4.2, p. 167] holds.

Second, we show \( \mathcal{L}_X \) is a normal lattice on \( X \). Let \( A, B \in \mathcal{L} \) where \( A \cap B \cap X = \emptyset \). Then \( \text{cl}_T (A \cap X) \cap \text{cl}_T (B \cap X) = \emptyset \) by hypothesis.

Using 2.1, select \( A', B' \in \mathcal{L} \) such that \( \text{cl}_T (A \cap X) \subset A' \), \( \text{cl}_T (B \cap X) \subset B' \) and \( A' \cap B' = \emptyset \). Since \( \mathcal{L} \) is a normal base on \( T \), select \( C, D \in \mathcal{L} \) such that \( A' \subset T - C \), \( B' \subset T - D \) and \( C \cup D = T \). Since \( \text{cl}_T (A \cap X) \subset T - C \) and \( \text{cl}_T (B \cap X) \subset T - D \), then we have \( A \cap X \subset X - (C \cap X) \), \( B \cap X \subset X - (D \cap X) \), \( (C \cap X) \cup (D \cap X) = X \) and \( C \cap X, D \cap X \in \mathcal{L}_X \). Hence, \( \mathcal{L}_X \) is a normal lattice on \( X \).

Using the above techniques, conditions (ii) and (iii) of Brooks' theorem [2, Theorem 4.2] are similarly verified. Hence, \( T = \omega \mathcal{L}_X \).

Since the converse holds immediately from Brooks' theorem, the proof is complete.

3. Dense Wallman bases. Let \( \mathcal{L} \) be a lattice on \( T \). For \( S \subset T \), define \( \mathcal{L}_S = \{ A \cap S \mid A \in \mathcal{L} \} \). Also, \( B \) is a co-\( \mathcal{L} \) set in \( T \) means \( B \) is the complement of an \( \mathcal{L} \)-set; i.e., there is an \( \mathcal{L} \)-set \( A \) such that \( B = T - A \) (denote \( B = \text{co}_T A \)).

3.1. Theorem. Let \( \mathcal{L} \) be a Wallman base on the space \( T \). Then \( \mathcal{L}_X \) is a Wallman base for each subspace \( X \) of \( T \) iff \( \mathcal{L}_Y \) is a Wallman base for each co-\( \mathcal{L} \) set \( Y \) in \( T \).

Proof. Suppose \( \mathcal{L}_Y \) is a Wallman base on \( Y \) for each co-\( \mathcal{L} \) set \( Y \) in \( T \). Let \( X \) be a subspace of \( T \). Then \( \mathcal{L}_X \) is a lattice on \( X \) which is a base for the closed subsets of \( X \). It remains to show that \( \mathcal{L}_X \) is normal. Let \( A, B \in \mathcal{L}_X \) such that \( A \cap B = \emptyset \). Select \( A_1, B_1 \in \mathcal{L} \) such that \( A = A_1 \cap X \) and \( B = B_1 \cap X \). Then \( X \subset S = \text{co}_T(A_1 \cap B_1) \). Let \( A_2 = A_1 \cap S \) and \( B_2 = B_1 \cap S \). Then \( A_2 \cap B_2 = \emptyset \). Since \( \mathcal{L}_S \) is a Wallman base on \( S \), select \( C_1, D_1 \in \mathcal{L} \) such that \( C = C_1 \cap S, D = D_1 \cap S \in \mathcal{L}_S \) where \( A_2 \subset \text{co}_S C \).
$B_2 \subseteq \cos D$ and $C \cup D = S$. Now $\cos G_1 \cap \cos T_1 \cap S = \emptyset$ since $(\cos E) \cap S = \cos (E \cap S)$ for each $E \in \mathcal{C}$. Thus $A \subseteq \cos (C_1 \cap X)$, $B \subseteq \cos (D_1 \cap X)$ and $(C_1 \cap X) \cup (D_1 \cap X) = X$. Hence, $\mathcal{L}_X$ is a normal base on $X$.

Similarly we obtain

3.2. Theorem. Let $\mathcal{L}$ be a Wallman base on the space $T$. Then $\mathcal{L}_X$ is a Wallman base for each dense subspace $X$ of $T$ iff $\mathcal{L}_Y$ is a Wallman base for each dense co-$\mathcal{L}$ set $Y$ in $T$.

3.3. Theorem. Let $T$ be a compact space and $\mathcal{L}$ a Wallman base on $T$. Then $T = \omega \mathcal{L}_X$ for each dense subspace $X$ of $T$ iff $T = \omega \mathcal{L}_Y$ for each dense co-$\mathcal{L}$ set $Y$ in $T$.

Proof. Suppose $T = \omega \mathcal{L}_Y$ for each dense co-$\mathcal{L}$ set $Y$ in $T$ and let $X$ be a dense subspace of $T$. We apply 2.4. Let $A, B \in \mathcal{L}_X$ such that $A \cap B = \emptyset$. Select $A_1, B_1 \in \mathcal{L}$ such that $A = A_1 \cap X$ and $B = B_1 \cap X$. Then $Y = \cos (A_1 \cap B_1)$ is a dense co-$\mathcal{L}$ set in $T$. Define $A_2 = A_1 \cap Y$ and $B_2 = B_1 \cap Y$. Since $A_2 \cap B_2 = \emptyset$ and $T = \omega \mathcal{L}_Y$, then $\cl T A_2 \cap \cl T B_2 = \emptyset$ by 2.4. But $A \subseteq \cl T A_2$ and $B \subseteq \cl T B_2$; so, $\cl T A \cap \cl T B = \emptyset$. Hence, $T = \omega \mathcal{L}_X$ by 2.4.

4. Compact $\mathcal{F}$-spaces. We now apply the previous theory to show that every compact $\mathcal{F}$-space (i.e., disjoint cozero sets are completely separated [GJ]) is a $\omega$-compactification of each of its dense subspaces. The ring of all continuously extendable functions from a space $X$ to an extension $T$ is denoted by $E(X, T)$.

4.1. Theorem. Let $T$ be a compact $\mathcal{F}$-space. Then $T$ is a $\omega$-compactification of each of its dense subspaces.

Proof. We apply 3.3. Now $Z[T]$ is a Wallman base on $T$. Let $Y$ be a dense cozero set of $T$. Then $C^*(Y) = E(Y, T)$ [GJ, 14.25 (5)] and so $T = \cl T Y = \beta Y$ [GJ, 6.5 (II)]. Also $\beta Y = \omega Z(Y)$ [GJ, 6.5], so $T = \omega Z(Y)$. And $Z(Y) = \{Z \cap Y \mid Z \in Z(T)\}$ since $Y$ is $\omega$-embedded in $T$ [5, Theorem 3]. By 3.3, $T = \omega Z(X)$ for each dense subspace $X$ in $T$. This completes the proof.

We remark that if $X$ is $\sigma$-compact and locally compact, then $\beta X - X$ is a compact $\mathcal{F}$-space [GJ, 14.27]. So $\beta R - R$ and $\beta N - N$ are $\omega$-compactifications of each of their dense subspaces, where $R$ denotes the reals and $N$ the positive integers with their usual topologies. Also if $Y$ is a dense cozero set of a compact space $T$, then $\beta Y - Y$ is a $\omega$-compactification of each of its dense subspaces [GJ, 14(0.2)].

5. Orderable compact spaces.

5.1. Theorem. Let $T$ be an ordered compact space. Then $T$ is a Wall-
man compactification of each of its dense subspaces.

Proof. Let \( \leq \) be an order on \( T \) compatible with the topology for \( T \). Let \( X \) be a dense subspace of \( T \). The left endpoint \( a \) of the interval \([a, b]\) in \( T \) is admissible means either \( a \in X \) or \([a, b]\) is a neighborhood of \( a \) in \( T \). The right endpoint \( b \) of the interval \([a, b]\) in \( T \) is admissible means either \( b \in X \) or \([a, b]\) is a neighborhood of \( b \) in \( T \). Let \( \mathcal{L} \) be the lattice on \( T \) of finite unions of closed intervals with admissible endpoints. Clearly \( X \) is \( \mathcal{L} \)-dense. By 2.2 and [1, Theorem 3] it then suffices to show that \( \mathcal{L} \) is a base for the closed sets in \( T \). Let \( F \) be a closed subset of \( T \) and \( p \in T - F \).

Case 1. \( p < t \) for each \( t \in F \). Let \( t' = \inf F \). If \( t' \in X \), then \([t', 1_T] \subseteq \mathcal{L} \) (\( 1_T = \sup T \)), \( F \subseteq [t', 1_T] \) and \( p \notin [t', 1_T] \). Suppose \( t' \notin X \). If \( x \in (p, t') \cap X \), then \([x, 1_T] \subseteq \mathcal{L} \), \( F \subseteq [x, 1_T] \) and \( p \notin [x, 1_T] \). If \( (p, t') \cap X = \emptyset \), then \([t', 1_T] \) is a \( T \)-neighborhood of \( t' \). So \([t', 1_T] \subseteq \mathcal{L} \), \( F \subseteq [t', 1_T] \) and \( p \in [t', 1_T] \).

Case 2. \( t < p \) for each \( t \in F \). Let \( t' = \sup F \) and proceed as in Case 1 obtaining \( q \in [t', p) \) such that \([0_T, q] \subseteq \mathcal{L} \) (\( 0_T = \inf T \)), \( F \subseteq [0_T, q] \) and \( p \notin [0_T, q] \).

Case 3. There exists \( q_1, q_2 \in F \) such that \( q_1 < p < q_2 \). Let \( F_1 = \{ t \in F \mid p < t \} \) and \( F_2 = \{ t \in F \mid t < p \} \). Then \( F_1, F_2 \) are disjoint closed subsets of \( T \) and \( F_1 \cup F_2 = F \). Apply Case 1 to \( p \) and \( F_1 \) obtaining \( t_1 \in T \) such that \([t_1, 1_T] \subseteq \mathcal{L} \), \( F_1 \subseteq [t_1, 1_T] \) and \( p \notin [t_1, 1_T] \). Apply Case 2 to \( F_2 \) and \( p \) obtaining \( t_2 \in T \) such that \([0_T, t_2] \subseteq \mathcal{L} \), \( F_2 \subseteq [0_T, t_2] \) and \( p \notin [0_T, t_2] \). Then \([0_T, t_2] \cup [t_1, 1_T] \subseteq \mathcal{L} \), \( F \subseteq [0_T, t_2] \cup [t_1, 1_T] \) and \( p \notin [0_T, t_2] \cup [t_1, 1_T] \). Hence, \( \mathcal{L} \) is a base for the closed subsets of \( T \). This completes the proof.

5.2. Theorem. Let \( X \) be an ordered space and let \( T \) be the maximal ordered compactification of \( X \). Then \( T \in zX \).

Proof. For the existence of \( T \), see [6]. \( T \) has the following property: if \( t \in T - X \), then there exists \( Y \subseteq X \) such that either \( t = \sup Y \) or \( t = \inf Y \), but never both. Let \( \mathcal{L} \) be the lattice on \( T \) of finite unions of sets of the form \( f^{-1}(a) \) where \( f : T \to [0, 1] \) is continuous and increasing, \( a \in [0, 1] \) and if \( t \in f^{-1}(a) \cap (T - X) \), then \( f^{-1}(a) \) is a neighborhood of \( t \) in \( T \). Using the method of the previous theorem, it is straightforward to show that \( \mathcal{L} \) is a Wallman base on \( T \) and that \( X \) is \( \mathcal{L} \)-dense. By [1, Theorem 3], \( T = wL_X \). Hence, \( T \in zX \) since \( \mathcal{L} \subseteq Z(T) \).

Although we have shown that the maximal ordered compactification of an ordered space is a \( z \)-compactification, we have not been able to show this for arbitrary ordered compactifications.
REFERENCES


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