ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

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ABSTRACT. Taking the start from an earlier result on the absolute harmonic summability factors due to S. N. Lal, we obtain in this paper suitable factors $\{\epsilon_n\}$ so that the series $\sum a_n\epsilon_n$ may be summable $|N, p_n|$, whenever the series $\sum a_n$ is summable $|C, 1|$.

1. Let $\{S_n\}$ denote the nth partial sum of the series $\sum a_n$. Let $\{p_n\}$ be a sequence of real numbers and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n, \quad P_{-1} = p_{-1} = 0.$$

The sequence to sequence transformation,

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}S_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} P_{n-\nu}a_\nu \qquad (P_n \neq 0), \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation [6], that is,

$$\sum_n |t_n - t_{n-1}| < \infty.$$

In the special case in which $p_n = 1/(n+1)$, the summability $|N, p_n|$ is the same as absolute harmonic summability.

Also, when

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha \geqq 0), \tag{1.2}$$

the Nörlund mean reduces to the familiar (C, α) mean. Thus the summability $|N, p_n|$, when p_n is given by (1.2) is the same as the summability $|C, \alpha|$.

2. Throughout the present paper we write

$$B_n = \sum_{\nu=1}^{n} \nu a_\nu, \qquad \Delta\chi_n = \chi_n - \chi_{n+1},$$

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and C to denote an absolute constant not necessarily the same at each occurrence.

3. In a paper in this journal Lal [4] established the following

THEOREM A. If $B_n = O(n)$, *then the series* $\sum a_n \log(n+1)\lambda n/n$, *where* $\{\lambda_n\}$ *is a convex sequence, such that* $\sum n^{-1}\lambda_n$ *is convergent, is absolutely harmonic summable.*

With the help of this theorem Lal obtained a result on absolute harmonic summability factors of Fourier series (see [4]). These results were extended by several workers such as Bhatt [2], Mohapatra, Das and Srivastava [8], Mehrotra [7], Ahmad [1], and others.

In this paper we establish the following absolute summability factor theorem, which when combined with a known result yields a result more general than Theorem A (see §6, Theorem 2).

THEOREM 1. *Let* $p_0 > 0$ *and* p_n *be a nonnegative and nonincreasing sequence. If a series* $\sum a_n$ *is summable* $|C, 1|$, *and a sequence* $\{\epsilon_n\}$ *is such that,*

$$n\epsilon_n = O(P_n), \tag{3.1}$$

and

$$n\Delta\epsilon_n = O(1), \tag{3.2}$$

as $n \to \infty$, *then the series* $\sum a_n\epsilon_n$ *is summable* $|N, p_n|$.

It is interesting to note that the following form of a result due to Kogbetliantz[1] [3] follows as a corollary of our theorem.

COROLLARY. *If a series* $\sum a_n$ *is summable* $|C, 1|$, *then the series* $\sum a_n\epsilon_n$ *is summable* $|C, \beta|$ $(0 < \beta \leqq 1)$,

where

$$\epsilon_n = 1/(n+1)^{1-\beta}.$$

4. We require the following lemma to establish our theorem.

LEMMA [1]. *If* $p_0 > 0$ *and* p_n *is a nonnegative and nonincreasing sequence, then for* $\nu \geqq 1$

$$\sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \leqq \frac{C}{\nu}; \tag{4.1}$$

$$\sum_{n=\nu}^{\infty} \frac{p_n(P_n - P_{n-\nu})}{P_n P_{n-1}} \leqq C, \tag{4.2}$$

[1] In 1952 Peyerimhoff [9], gave a simpler proof of Kogbetliantz's result.

$$\sum_{n=\nu}^{\infty} \frac{|\Delta_n p_{n-\nu-1}|}{P_{n-1}} \leqq \frac{C}{P_\nu} + \frac{C}{\nu}, \tag{4.3}$$

and

$$\sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \leqq C. \tag{4.4}$$

5. Proof of the theorem. Let τ_n denote the nth Nörlund mean of the series $\sum a_n\epsilon_n$. Then by definition,

$$\tau_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}\sum_{\mu=0}^{\nu} a_\mu\epsilon_\mu = \frac{1}{P_n}\sum_{\nu=0}^{n} P_{n-\nu}a_\nu\epsilon_\nu,$$

so that

$$\begin{aligned}
\tau_n - \tau_{n-1} &= \frac{1}{P_nP_{n-1}}\sum_{\nu=1}^{n}(P_np_{n-\nu} - P_{n-\nu}p_n)\epsilon_\nu a_\nu \\
&= \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n}(P_n - P_{n-\nu})\epsilon_\nu a_\nu \\
&\quad + \frac{1}{P_{n-1}}\sum_{\nu-1}^{n}(p_{n-\nu} - p_n)\epsilon_\nu a_\nu \\
&= \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}\Delta_\nu\left\{(P_n - P_{n-\nu})\frac{\epsilon_\nu}{\nu}\right\}B_\nu \\
&\quad + \frac{p_n}{P_nP_{n-1}}\left\{(P_n - P_0)\frac{\epsilon_n}{n}\right\}B_n \\
&\quad + \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}\Delta_\nu\left\{(p_{n-\nu} - p_n)\frac{\epsilon_\nu}{\nu}\right\}B_\nu \\
&\quad + \frac{1}{P_{n-1}}\left\{(p_0 - p_n)\frac{\epsilon_n}{n}\right\}B_n \\
&= \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n}\Delta_\nu\left\{(P_n - P_{n-\nu})\frac{\epsilon_\nu}{\nu}\right\}B_\nu \\
&\quad + \frac{1}{P_{n-1}}\sum_{\nu=1}^{n}\Delta_\nu\left\{(p_{n-\nu} - p_n)\frac{\epsilon_\nu}{\nu}\right\}B_\nu,
\end{aligned}$$

and therefore

$$|\tau_n - \tau_{n-1}| \leqq \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\epsilon_\nu}{\nu} \right\} \right| |B_\nu| + \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\epsilon_\nu}{\nu} \right\} \right| |B_\nu| .$$

Thus to establish the theorem we have to show that

$$(5.1) \quad \Sigma_1 = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\epsilon_\nu}{\nu} \right\} \right| |B_\nu| < \infty,$$

and

$$(5.2) \quad \Sigma_2 = \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\epsilon_\nu}{\nu} \right\} \right| |B_\nu| < \infty.$$

Before proceeding to the proof of (5.1) and (5.2), we observe that, under the hypothesis (3.1) of our theorem $\epsilon_n = O(P_n/n) = O(1)$, as $n \to \infty$ since P_n/n is monotonic nonincreasing [5]. We now have

$$\begin{aligned}
\Sigma_1 = {} & O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) |\epsilon_\nu| \frac{|B_\nu|}{\nu^2} \\
& + O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) |\Delta\epsilon_\nu| \frac{|B_\nu|}{(\nu+1)} \\
& + O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} p_{n-\nu} |\epsilon_{\nu+1}| \frac{|B_\nu|}{(\nu+1)} \\
& + O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| |B_\nu|}{\nu^2} \sum_{n=\nu}^{\infty} \frac{p_n(P_n - P_{n-\nu})}{P_n P_{n-1}} \\
& + O(1) \sum_{\nu=1}^{\infty} \frac{|\Delta\epsilon_\nu| |B_\nu|}{(\nu+1)} \sum_{n=\nu}^{\infty} \frac{p_n(P_n - P_{n-\nu})}{P_n P_{n-1}} \\
& + O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_{\nu+1}| |B_\nu|}{(\nu+1)} \sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \\
= {} & O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| |B_\nu|}{\nu^2} + O(1) \sum_{\nu=1}^{\infty} \frac{|\Delta\epsilon_\nu| |B_\nu|}{\nu} \\
= {} & O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| |B_\nu|}{\nu^2} + O(1) \sum_{\nu=1}^{\infty} \frac{\nu |\Delta\epsilon_\nu| |B_\nu|}{\nu^2} \\
= {} & O(1) \sum_{\nu=1}^{\infty} \frac{|B_\nu|}{\nu^2} \\
= {} & O(1)
\end{aligned}$$

by (4.1), (4.2) of the lemma and under the hypotheses of the theorem (since the series $\sum a_n$ is summable $|C, 1|$, $\sum |B_n|/n^2 < \infty$). Again,

$$\begin{aligned}
\Sigma_2 &= O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \, |\epsilon_\nu| \, \frac{|B_\nu|}{\nu^2} \\
&\quad + O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \, |\Delta\epsilon_\nu| \, \frac{|B_\nu|}{(\nu+1)} \\
&\quad + O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} |\Delta_n p_{n-\nu-1}| \, |\epsilon_{\nu+1}| \, \frac{|B_\nu|}{(\nu+1)} \\
&= O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| \, |B_\nu|}{\nu^2} \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \\
&\quad + O(1) \sum_{\nu=1}^{\infty} \frac{|\Delta\epsilon_\nu| \, |B_\nu|}{(\nu+1)} \sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \\
&\quad + O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_{\nu+1}| \, |B_\nu|}{(\nu+1)} \sum_{n=\nu}^{\infty} \frac{|\Delta_n p_{n-\nu-1}|}{P_{n-1}} \\
&= O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| \, |B_\nu|}{\nu^2} + O(1) \sum_{\nu=1}^{\infty} \frac{|\Delta\epsilon_\nu| \, |B_\nu|}{\nu} \\
&\quad + O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_{\nu+1}| \, |B_\nu|}{(\nu+1)P_\nu} + O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_{\nu+1}| \, |B_\nu|}{\nu(\nu+1)} \\
&= O(1) \sum_{\nu=1}^{\infty} \frac{|\epsilon_\nu| \, |B_\nu|}{\nu^2} + O(1) \sum_{\nu=1}^{\infty} \frac{\nu |\Delta\epsilon_\nu| \, |B_\nu|}{\nu^2} \\
&\quad + O(1) \sum_{\nu=1}^{\infty} \frac{\nu |\epsilon_{\nu+1}| \, |B_\nu|}{\nu(\nu+1)P_\nu} \\
&= O(1) \sum_{\nu=1}^{\infty} \frac{|B_\nu|}{\nu^2} \\
&= O(1)
\end{aligned}$$

by (4.3), (4.4) of the lemma and under the hypotheses of the theorem.

Combining the estimates of Σ_1 and Σ_2, we see that the theorem is completely established.

6. Prasad and Bhatt [10] established the following

THEOREM B. *If $\{\lambda_n\}$ is a convex sequence, such that $\sum n^{-1}\lambda_n$ is convergent, and*

$$\frac{1}{n}\sum_{\nu=1}^{n}\frac{|B_\nu|}{\nu} = O(\log n)^k \qquad (k \geqq 0), \tag{6.1}$$

then the series $\sum\{\log (n+1)\}^{-k}\lambda_n a_n$ *is summable* $|C, 1|$.

If we combine the above result with Theorem 1 we get the following

THEOREM 2. *If* $\{\lambda_n\}$ *is a convex sequence, such that* $\sum n^{-1}\lambda_n$ *is convergent, and if* (6.1) *holds then the series* $\sum(a_n\lambda_n\epsilon_n/\{\log (n+1)\}^k)$ *is summable* $|N, p_n|$, *provided that*

$$n\epsilon_n = O(P_n), \quad \text{and} \quad n\Delta\epsilon_n = O(1),$$

as $n\to\infty$.

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