ON THE ABSOLUTE NÖRLUND SUMMABILITY 
FACTORS OF INFINITE SERIES

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Abstract. Taking the start from an earlier result on the absolute harmonic summability factors due to S. N. Lal, we obtain in this paper suitable factors \( \{e_n\} \) so that the series \( \sum a_n e_n \) may be summable \( |N, p_n| \), whenever the series \( \sum a_n \) is summable \( |C, 1| \).

1. Let \( \{S_n\} \) denote the \( n \)th partial sum of the series \( \sum a_n \). Let \( \{p_n\} \) be a sequence of real numbers and let us write

\[
P_n = p_0 + p_1 + p_2 + \cdots + p_n, \quad P^{-1} = p^{-1} = 0.
\]

The sequence to sequence transformation,

\[
(1.1) \quad t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_{n-r} S_r = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r} a_r \quad (P_n \neq 0),
\]

defines the sequence \( \{t_n\} \) of Nörlund means of the sequence \( \{S_n\} \), generated by the sequence of coefficients \( \{p_n\} \).

The series \( \sum a_n \) is said to be absolutely summable \( (N, p_n) \), or summable \( |N, p_n| \), if the sequence \( \{t_n\} \) is of bounded variation [6], that is,

\[
\sum_n |t_n - t_{n-1}| < \infty.
\]

In the special case in which \( p_n = 1/(n + 1) \), the summability \( |N, p_n| \) is the same as absolute harmonic summability.

Also, when

\[
(1.2) \quad p_n = \left( \frac{n + \alpha - 1}{\alpha - 1} \right) = \frac{\Gamma(n + \alpha)}{\Gamma(n + 1) \Gamma(\alpha)} \quad (\alpha \geq 0),
\]

the Nörlund mean reduces to the familiar \( (C, \alpha) \) mean. Thus the summability \( |N, p_n| \), when \( p_n \) is given by (1.2) is the same as the summability \( |C, \alpha| \).

2. Throughout the present paper we write

\[
B_n = \sum_{\nu=1}^{n} \nu a_{\nu}, \quad \Delta x_n = x_n - x_{n+1},
\]

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and $C$ to denote an absolute constant not necessarily the same at each occurrence.

3. In a paper in this journal Lal [4] established the following

**Theorem A.** If $B_n = O(n)$, then the series $\sum a_n \log(n+1)/n$, where $\{\lambda_n\}$ is a convex sequence, such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely harmonic summable.

With the help of this theorem Lal obtained a result on absolute harmonic summability factors of Fourier series (see [4]). These results were extended by several workers such as Bhatt [2], Mohapatra, Das and Srivastava [8], Mehrotra [7], Ahmad [1], and others.

In this paper we establish the following absolute summability factor theorem, which when combined with a known result yields a result more general than Theorem A (see §6, Theorem 2).

**Theorem 1.** Let $p_0 > 0$ and $p_n$ be a nonnegative and nonincreasing sequence. If a series $\sum a_n$ is summable $C$, $1$, and a sequence $\{\epsilon_n\}$ is such that,

3.1) $n\epsilon_n = O(P_n)$,

and

3.2) $n\Delta\epsilon_n = O(1)$,

as $n \to \infty$, then the series $\sum a_n\epsilon_n$ is summable $N$, $p_n$.

It is interesting to note that the following form of a result due to Kogbetliantz \[3\] follows as a corollary of our theorem.

**Corollary.** If a series $\sum a_n$ is summable $C$, $1$, then the series $\sum a_n\epsilon_n$ is summable $C, \beta$ $(0 < \beta \leq 1)$,

where

$\epsilon_n = 1/(n + 1)^{1-\beta}$.

4. We require the following lemma to establish our theorem.

**Lemma [1].** If $p_0 > 0$ and $p_n$ is a nonnegative and nonincreasing sequence, then for $\nu \geq 1$

\[4.1\] $\sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \leq C$, $\nu$

\[4.2\] $\sum_{n=\nu}^{\infty} \frac{p_n(P_n - P_{n-\nu})}{P_n P_{n-1}} \leq C$.

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1 In 1952 Peyerimhoff [9], gave a simpler proof of Kogbetliantz's result.
5. Proof of the theorem. Let \( \tau_n \) denote the \( n \)th Nörlund mean of the series \( \sum a_n \varepsilon_n \). Then by definition,

\[
\tau_n = \frac{1}{P_n} \sum_{r=0}^{n} \varphi_n^r \sum_{\mu=0}^{r} a_{\mu} \varepsilon_{\mu} = \frac{1}{P_n} \sum_{r=0}^{n} \sum_{\mu=0}^{r} P_{n-r} a_{r} \varepsilon_{r},
\]

so that

\[
\tau_n - \tau_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{r=1}^{n} (P_n \varphi_n^r - P_{n-r} \varphi_n) \varepsilon_r a_r
\]

\[= \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n} (P_n - P_{n-r}) \varepsilon_r a_r
\]

\[+ \frac{1}{P_{n-1}} \sum_{r=1}^{n} (\varphi_n^r - \varphi_n) \varepsilon_r a_r
\]

\[= \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n-1} \Delta_r \left\{ (P_n - P_{n-r}) \varepsilon_r \right\} B_r
\]

\[+ \frac{\varphi_n}{P_n P_{n-1}} \left\{ (P_n - P_0) \frac{\varepsilon_n}{n} \right\} B_n
\]

\[+ \frac{1}{P_{n-1}} \sum_{r=1}^{n-1} \Delta_r \left\{ (\varphi_n^r - \varphi_n) \frac{\varepsilon_r}{n} \right\} B_r
\]

\[= \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n-1} \Delta_r \left\{ (P_n - P_{n-r}) \frac{\varepsilon_r}{n} \right\} B_r
\]

\[+ \frac{1}{P_{n-1}} \left\{ (P_0 - \varphi_n) \frac{\varepsilon_n}{n} \right\} B_n
\]

and therefore

\[
\sum_{n=r}^{\infty} \left| \frac{\Delta_n p_{n-r-1}}{P_{n-1}} \right| \leq \frac{C}{P_r} + \frac{C}{r},
\]

and

\[
\sum_{n=r}^{\infty} \left| \frac{p_{n-r} - p_n}{P_{n-1}} \right| \leq C.
\]
Thus to establish the theorem we have to show that

\[(5.1) \quad \sum_{n=1}^{\infty} \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n} \Delta_r \left\{ \left( P_n - P_{n-r} \right) \frac{\epsilon_r}{\nu} \right\} \left| B_r \right| < \infty, \]

and

\[(5.2) \quad \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{r=1}^{n} \Delta_r \left\{ \left( P_{n-r} - P_n \right) \frac{\epsilon_r}{\nu} \right\} \left| B_r \right| < \infty. \]

Before proceeding to the proof of (5.1) and (5.2), we observe that, under the hypothesis (3.1) of our theorem \( \epsilon_n = O(P_n/n) = O(1) \), as \( n \to \infty \) since \( P_n/n \) is monotonic nonincreasing [5]. We now have

\[
\sum_1 = O(1) \sum_{n=1}^{\infty} \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n} \left( P_n - P_{n-r} \right) | \epsilon_r | \left| \frac{B_r}{\nu^2} \right| \\
+ O(1) \sum_{n=1}^{\infty} \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n} \left( P_n - P_{n-r} \right) \Delta \epsilon_r \left| \frac{B_r}{(\nu + 1)} \right| \\
+ O(1) \sum_{n=1}^{\infty} \frac{\varphi_n}{P_n P_{n-1}} \sum_{r=1}^{n} \epsilon_{r+1} \left| \frac{B_r}{(\nu + 1)} \right| \\
+ O(1) \sum_{n=1}^{\infty} \frac{\epsilon_r \left| B_r \right|}{\nu^2} \sum_{n=1}^{\infty} \frac{\varphi_n (P_n - P_{n-r})}{P_n P_{n-1}} \\
+ O(1) \sum_{n=1}^{\infty} \frac{\Delta \epsilon_r \left| B_r \right|}{(\nu + 1)} \sum_{n=1}^{\infty} \frac{\varphi_n (P_n - P_{n-r})}{P_n P_{n-1}} \\
+ O(1) \sum_{n=1}^{\infty} \frac{\epsilon_{r+1} \left| B_r \right|}{(\nu + 1)} \sum_{n=1}^{\infty} \frac{\varphi_n P_{n-r}}{P_n P_{n-1}} \\
= O(1) \sum_{r=1}^{\infty} \frac{| \epsilon_r | \left| B_r \right|}{\nu^2} + O(1) \sum_{r=1}^{\infty} \frac{| \Delta \epsilon_r | \left| B_r \right|}{\nu} \\
= O(1) \sum_{r=1}^{\infty} \frac{| \epsilon_r | \left| B_r \right|}{\nu^2} + O(1) \sum_{r=1}^{\infty} \frac{\nu | \Delta \epsilon_r | \left| B_r \right|}{\nu^2} \\
= O(1) \sum_{r=1}^{\infty} \frac{| B_r |}{\nu^2} \\
= O(1)
\]
by (4.1), (4.2) of the lemma and under the hypotheses of the theorem (since the series \( \sum a_n \) is summable \(| C, 1 |, \sum |B_n|/n^2 < \infty\). Again,

\[
\Sigma_2 = O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{r=1}^{n} (p_{n-r} - p_n) |\varepsilon_r| \frac{|B_r|}{\nu^2}
\]

\[
+ O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{r=1}^{n} |\Delta p_{n-r-1}| |\varepsilon_{r+1}| \frac{|B_r|}{\nu + 1}
\]

\[
+ O(1) \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{r=1}^{n} |\Delta p_{n-r-1}| |\varepsilon_{r+1}| \frac{|B_r|}{\nu + 1}
\]

= \(O(1)\) \sum_{r=1}^{\infty} \frac{|\varepsilon_r| |B_r|}{\nu^2} + \(O(1)\) \sum_{r=1}^{\infty} \frac{|\Delta \varepsilon_r| |B_r|}{\nu}

+ \(O(1)\) \sum_{r=1}^{\infty} \frac{|\varepsilon_{r+1}| |B_r|}{\nu + 1} \nu + 1

+ \(O(1)\) \sum_{r=1}^{\infty} \nu |\Delta \varepsilon_r| \frac{|B_r|}{\nu^2}

+ \(O(1)\) \sum_{r=1}^{\infty} \nu |\varepsilon_{r+1}| \frac{|B_r|}{\nu + 1}

= \(O(1)\) \sum_{r=1}^{\infty} \frac{|B_r|}{\nu^2}

= \(O(1)\)

by (4.3), (4.4) of the lemma and under the hypotheses of the theorem.

Combining the estimates of \(\Sigma_1\) and \(\Sigma_2\), we see that the theorem is completely established.

6. Prasad and Bhatt [10] established the following

**Theorem B.** If \(\{\lambda_n\}\) is a convex sequence, such that \(\sum n^{-1}\lambda_n\) is convergent, and
then the series \( \sum \{ \log (n+1) \}^{-k} \lambda_n a_n \) is summable \( |C, 1| \).

If we combine the above result with Theorem 1 we get the following

**Theorem 2.** If \( \{ \lambda_n \} \) is a convex sequence, such that \( \sum n^{-1} \lambda_n \) is convergent, and if (6.1) holds then the series \( \sum (a_n \lambda_n \epsilon_n / \{ \log (n+1) \}^k) \) is summable \( |N, p_n| \), provided that

\[
n \epsilon_n = O(P_n), \quad \text{and} \quad n \Delta \epsilon_n = O(1),
\]

as \( n \to \infty \).

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**References**


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