ON REGULAR MATRICES THAT INDUCE
THE GIBBS PHENOMENON

JOAQUIN BUSTOZ

Abstract. Let \( s = \{s_n(z)\} \) be a sequence of complex valued
functions defined in a subset \( D \) of the complex plane and suppose
that \( s_n(z) \) converges to \( f(z) \) for \( z \in D \). For \( z_0 \in \overline{D} \) let \( K(z_0; s) \) and
\( K(z_0; f) \) be the cores of \( s \) and \( f \) respectively. We say that \( s \) does not
have the Gibbs phenomenon at \( z_0 \) if \( K(z_0; s) \subseteq K(z_0; f) \). The regular
matrix \( A \) is said to induce the Gibbs phenomenon in \( s \) if \( K(z_0; s) \subseteq K(z_0; f) \)
but \( K(z_0; As) \supseteq K(z_0; f) \). We characterize those regular
matrices that induce the Gibbs phenomenon.

1. Let \( s = \{s_n(z)\} \) be a sequence of complex valued functions de-
   fined in a subset \( D \) of the complex plane. Denote the closure of \( D \) by
\( \overline{D} \), and let \( z_0 \in \overline{D} \). Define the sets

\[
G_n(z_0; s) = \{ w : w = s_k(z), k \geq n, z \in D, |z - z_0| < 1/n \}.
\]

Then set \( G(z_0; s) = \bigcap_{n=1}^{\infty} \overline{G_n(z_0; s)} \). It is easy to see that \( G(z_0; s) \) is the
set of complex numbers \( w \) of the form \( w = \lim s_{n(k)}(z_k) \) where \( n(1) < n(2) < \cdots \)
and \( z_k \to z_0 \). When \( s \) is the sequence of partial sums of a
real valued Fourier series the set \( G(z_0; s) \) is called the Gibbs set of \( s \) at
\( z_0 \). We will use the same terminology for \( G(z_0; s) \) when \( s \) is any se-
quence. In the case of Fourier series \( G(z_0; s) \) is connected as a con-
sequence of the Riemann-Lebesgue Theorem. We will show that
\( G(z_0; s) \) is connected in the plane under certain conditions on \( s \).

Let the closed convex hull of \( G_n(z_0; s) \) be denoted by \( \overline{G_n(z_0; s)} \),
and set \( K(z_0; s) = \bigcap_{n=1}^{\infty} \overline{G_n(z_0; s)} \). The closed convex set \( K(z_0; s) \)
is called the core of \( s \) at \( z_0 \). The Gibbs set \( G(z_0; f) \) and core \( K(z_0; f) \) of a
function \( f(z) \) are the Gibbs set and core of the sequence \( f_n(z) \equiv f(z) \).

\( G(z_0; f) \) is simply the cluster set of \( f \) at \( z_0 \). Suppose that \( s \) converges to
\( f(z) \) in \( D \). We will say that \( s \) does not exhibit the Gibbs phenomenon
at \( z_0 \in \overline{D} \) if \( K(z_0; s) \subseteq K(z_0; f) \). This reduces to the usual definition
when \( s \) is the partial sum sequence of a Fourier series. In §3 we will
use the core definition to investigate the effect of regular matrix trans-
formation on the Gibbs phenomenon.

The concept of core is due to K. Knopp who defined it for se-
quences of complex numbers. There is a large body of literature con-
cerning cores of sequences and matrix summability. An account of
this is given in [2] along with an extensive bibliography.

Received by the editors June 20, 1969.

AMS Subject Classifications. Primary 4020, 4031; Secondary 4220.

Key Words and Phrases. Core, Gibbs phenomenon, regular matrix, Gibbs set.
2. In this section we will suppose that $D$ is a locally connected subset of the complex plane. We will consider sequences that satisfy the following condition.

(a) There exists $\{\xi_k\} \subset D$, $\xi_k \rightarrow z_0 \in \overline{D}$, such that

$$\lim_{n \rightarrow \infty} \left[ s_{n+1}(\xi_k) - s_n(\xi_k) \right] = 0 \quad \text{for } k = 1, 2, \ldots.$$ 

**Lemma.** Let $s = \{s_n(z)\}$ be a sequence of continuous functions satisfying (a). Let the nonempty sets $B_1$ and $B_2$ satisfy the following conditions.

(i) $G(z_0; s) \subset \text{int } B_1 \cup \text{int } B_2$, where $\text{int } B_i$ is the interior of $B_i$.

(ii) $G(z_0; s) \cap \text{int } B_i \neq \emptyset$, $i = 1, 2$.

(iii) There exist $\delta > 0$ and $N$ such that $s_n(z) \in B_1 \cup B_2$ whenever $z \in D$ with $|z - z_0| < \delta$ and $n > N$.

Then $d(B_1, B_2) = \inf \{|w_1 - w_2| : w_i \in B_i\} = 0$.

**Proof.** Suppose that $d(B_1, B_2) = r > 0$. If $s_n(w) \in B_1$ with $|z_0 - w| \leq \delta$ then $s_n(z) \in B_1$ for every $z \in D$ with $|z - z_0| \leq \delta$, since $s_n$ is continuous and $B_1, B_2$ are separated. By condition (a) there exists $\xi_i \in \{\xi_k\}$ such that $|\xi_i - z_0| < \delta$ and there exists $N$ such that $n \geq N$ implies $|s_n(\xi_i) - s_{n-1}(\xi_i)| < r/2$. Suppose that $s_N(\xi_i) \in B_1$. Since $\text{int } B_2 \cap G(z_0; s) \neq \emptyset$ there is a first index $M > N$ such that $s_M(\xi_i) \in B_2$. Then $s_{M-1}(\xi_i) \in B_1$ and hence $|s_M(\xi_i) - s_{M-1}(\xi_i)| \geq r$. This contradicts the previous inequality and the lemma is proved.

We will say that a set $Q$ is connected in the extended complex plane if in any separation $Q = Q_1 \cup Q_2$, both $Q_1$ and $Q_2$ are unbounded.

**Theorem 1.** Let $s$ be a sequence of continuous functions satisfying condition (a). Then $G(z_0; s)$ is connected in the extended plane. If $s$ is uniformly bounded in $D$ then $G(z_0; s)$ is connected.

**Proof.** We will prove the second statement. The proof of the first is similar and is omitted. Suppose that $G(z_0; s)$ is not connected. Then there exist nonempty compact sets $Q_1$ and $Q_2$ such that $G(z_0; s) = Q_1 \cup Q_2$ and $d(Q_1, Q_2) = r_1 > 0$. Further, there exist compact sets $B_1$ and $B_2$ such that $\text{int } B_i \supset Q_i$ and $0 < d(B_1, B_2) = r_2 < r_1$. But then $B_1$ and $B_2$ satisfy the conditions of the lemma, and thus $d(B_1, B_2) = 0$. This contradiction establishes the result.

If $s$ is real valued it is not necessary to distinguish between the bounded and unbounded case. It is perhaps worthwhile to state this as

**Theorem 2.** Let $s$ be a sequence of continuous real functions that satisfy (a). Then $G(z_0; s)$ is connected.
The sequence \( \{e^{in}\} \) shows that (\( \alpha \)) is not necessary for connectedness. Examples of sequences that satisfy (\( \alpha \)) are sequences converging in \( D \), and partial sums of Fourier series.

3. Let \( A = (a_{nk}) \) be a regular summability matrix. We will write \( t = As \) for the transform sequence \( t_n(z) = \sum a_{nk} s_k(z) \). In this section we will consider the following situation. Suppose that \( s = \{s_n(z)\} \) converges pointwise to \( f(z) \) in \( D \). Suppose further that at \( z_0 \in \overline{D} \), \( K(z_0; s) \subseteq K(z_0; f) \), so that \( s \) does not have the Gibbs phenomenon at \( z \). Is it possible for \( As \) to show the phenomenon at \( z_0 \) even though \( s \) does not? We will show that this is indeed possible, and when this occurs we will say that \( A \) induces the Gibbs phenomenon in \( s \). We will use the notation \( \|A\| = \limsup(n \to \infty) \sum |a_{nk}| \).

**Theorem 3.** The regular matrix \( A \) induces the Gibbs phenomenon in some uniformly bounded sequence if and only if \( \|A\| > 1 \).

**Proof.** Since \( A \) is regular, \( \|A\| \geq 1 \). Suppose that \( \|A\| = 1 \), and that \( s \) is a uniformly bounded sequence converging to \( f \) with \( K(z_0; s) \subseteq K(z_0; f) \). Since \( \|A\| = 1 \) the Bounded Core Theorem [1] implies \( K(z_0; As) \subseteq K(z_0; s) \). Hence \( K(z_0; As) \subseteq K(z_0; f) \) and \( A \) does not induce the Gibbs phenomenon. To prove the theorem in the opposite direction we will show that if \( \|A\| > 1 \) then it is possible to construct a sequence with the desired property. The proof is an adaptation of a proof due to W. Hurwitz [3]. If \( \|A\| > 1 \) then there exist \( \lambda > 0 \) and sequences of natural numbers \( \{n(p)\} \), \( \{N(p)\} \) such that for \( p = 1, 2, \ldots \):

\[
\sum_{k=1}^{N(p)} |a_{n(p),k}| > 1 + 4\lambda, \quad \sum_{k=1}^{N(p)-1} |a_{n(p),k}| < \lambda, \quad \sum_{k=N(p)+1}^{\infty} |a_{n(p),k}| < \lambda.
\]

For the purpose of the construction we assume that the complement of \( \{N(p)\} \) in the natural numbers has the cardinality of the natural numbers. This assumption does not result in any loss of generality. Let \( f(x) = \exp i/x, x \neq 0, -1 \leq x \leq 1 \) and \( f(0) = 0 \).

Then \( K(0, f) \) is the closed unit disc. Now define the sequence \( s = \{s_k(x)\} \) as follows. We will write \( x_p = 1/n(p) \).

\[
s_k(0) = 0, \quad k = 1, 2, \ldots,
\]

\[
s_k(x) = \exp i/x \quad \text{if } x \neq 0 \text{ and } x \neq 1/N(p), \quad k = 1, 2, \ldots,
\]

\[
s_k(x_p) = \exp iN(p), \quad k \leq N(p - 1),
\]

\[
= (-1)^p \exp(-i \arg a_{n(p),k}), \quad \text{if } N(p - 1) < k \leq N(p),
\]

\[
= \exp iN(p), \quad k > N(p).
\]
Then \( s_n(x) \rightarrow f(x) \) and \( K(0; s) \) is the closed unit disc. Hence \( s \) does not exhibit the Gibbs phenomenon. Now

\[
\begin{align*}
    t_n(p)(x_p) &= \sum_k a_{n(p),k}S_k(x_p) \\
    &= \sum_{k=1}^{N(p-1)} a_{n(p),k}S_k(x_p) + \sum_{k=N(p-1)+1}^{N(p)} a_{n(p),k}S_k(x_p) \\
    &+ \sum_{k=N(p)+1}^{\infty} a_{n(p),k}S_k(x_p)
\end{align*}
\]

\[
\begin{align*}
    &= \exp iN(p) \sum_{k=1}^{N(p-1)} a_{n(p),k} - (-1)^p \sum_{k=1}^{N(p-1)} |a_{n(p),k}|
    \\
    &+ (-1)^p \sum_{k=1}^{N(p)} |a_{n(p),k}| + \exp iN(p) \sum_{k=N(p)+1}^{\infty} a_{n(p),k}.
\end{align*}
\]

Taking real parts:

\[
\begin{align*}
    (-1)^p \Re t_n(p)(x_p) &= \sum_{k=1}^{N(p)} |a_{n(p),k}| - \sum_{k=1}^{N(p-1)} |a_{n(p),k}|
    \\
    &+ (-1)^p \Re \exp iN(p) \sum_{k=1}^{N(p-1)} a_{n(p),k}
    \\
    &+ (-1)^p \Re \exp iN(p) \sum_{k=N(p)+1}^{\infty} a_{n(p),k}
    \\
    &= \sum_{k=1}^{N(p)} |a_{n(p),k}| - 2 \sum_{k=1}^{N(p-1)} |a_{n(p),k}|
    \\
    &+ \sum_{k=N(p)+1}^{\infty} |a_{n(p),k}|
    \\
    &> 1 + 4\lambda - 2\lambda - \lambda = 1 + \lambda.
\end{align*}
\]

Now \((-1)^p \Re t_n(p)(x_p) > 1 + \lambda\) implies that \( K(\varepsilon_0; As) \) must have points lying outside the unit circle. Hence \( K(0; f) \) does not contain \( K(0; As) \) and \( A \) induces the Gibbs phenomenon in \( s \).

It is possible to prove a theorem similar to Theorem 3 that holds for sequences of bounded functions. We must however restrict ourselves to row-finite regular matrices. The following terminology will be used. The row-finite matrix \( A \) will be said to be essentially real and nonnegative if the elements of \( A \) are real and nonnegative except possibly for finitely many rows or columns. A preliminary result will be required. This next theorem is an extension of the unbounded
core theorem for sequences of complex numbers [2, p. 148]. The proof is similar to that given in the reference and is omitted.

**Theorem 4.** Let $A$ be row-finite and regular. Then $K(z_0; A_s) \subseteq K(z_0; s)$ for every sequence of bounded functions if and only if $A$ is essentially real and nonnegative.

**Theorem 5.** Let $A$ be row-finite and regular. Then $A$ does not induce the Gibbs Phenomenon in any sequence of bounded functions if and only if $A$ is essentially real and nonnegative.

**Proof.** Let $s_n(z) \rightarrow f(z)$ and $K(z_0; s) \subseteq K(z_0; f)$. If $A$ is essentially real and nonnegative then $K(z_0; A_s) \subseteq K(z_0; f)$ by Theorem 4. Hence $A$ does not induce the Gibbs phenomenon. To prove the converse suppose first that $a_n(p), k(p) < 0$ where $n(1) < n(2) < \cdots$ and $k(1) < k(2) < \cdots$. Let $f(x) = 1/x$ for $0 < x \leq 1$ and $f(0) = 0$. Set $x_p = 1/n(p)$. If $k \notin \{k(p)\}$ set $s_k(x) = 1/x$ for $1/k \leq x \leq 1$ and $s_k(x) = 0$ for $0 \leq x < 1/k$. Set $s_{k(p)}(x_p) = -1/a_{n(p), k(p)}$; $s_{k(p)}(x) = 0$ for $0 \leq x < x_p$ and $s_{k(p)}(x) = 1/x$ for $x_p < x \leq 1$. Then $s = \{s_k(x)\}$ is a sequence of bounded functions converging to $f$, and $K(0, s) = K(0, f) = [0, \infty)$. But $t_{n_p}(x_p) = -1$ and hence $K(0, A_s)$ is not contained in $K(0, f)$. Thus $A$ induces the Gibbs phenomenon. On the other hand, suppose that $\text{Im } a_{n(p), k(p)} \neq 0$ for $p = 1, 2, \cdots$. Say that $\text{Im } a_{n(p), k(p)} < 0$. Let $f(x) = i/x$ for $0 < x \leq 1$ and $f(0) = 0$. If $k \notin \{k(p)\}$ then set $s_k(x) = i/x$ for $1/k \leq x \leq 1$; $s_k(x) = 0$ for $0 \leq x < 1/k$. Also set $s_{k(p)}(x_p) = i/\text{Im } a_{n(p), k(p)}$; $s_{k(p)}(x) = 0$ if $0 \leq x < x_p$, and $s_{k(p)}(x) = i/x$ if $x_p < x \leq 1$. Again $K(0, s) = K(0, f)$ each set consisting of the imaginary axis $iy$, $y \geq 0$. But $\text{Re } t_{n_p}(x_p) = -1$ and hence $K(0, A_s)$ is not contained in $K(0, f)$. This completes the proof.

4. It is of interest to ask to what extent Theorems 3 and 5 apply to Fourier series. Clearly if $\|A\| = 1$ then $A$ cannot induce the Gibbs phenomenon in uniformly bounded Fourier series. Also if $A$ is essentially real and nonnegative, then $A$ cannot induce the phenomenon in any Fourier series. But when $A$ does not satisfy either of the above conditions there does not necessarily exist a Fourier sequence in which $A$ induces the Gibbs phenomenon. Thus we can ask the following question: Does there exist a regular matrix $A$ and a Fourier series in which $A$ induces the Gibbs phenomenon?

We will answer this question affirmatively by proving a more general result. Previously we have considered $G(z_0; s)$ with $z_0 \in D$ where $D$ is the domain of definition of $s$. In this section we will take $z_0 \in D$ so that each $s_n$ is defined at $z_0$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem 6. Let $s = \{s_n(z)\}$ be a sequence of functions continuous and uniformly bounded in $D$. Let $z_0 \in D$ and let $G(z_0; s)$ contain more than one point. Then for every point $w$ there exists a regular matrix $A$ such that $w \in G(z_0; As)$.

Proof. We may take $z_0 = 0$. Our proof is based on the assertion that under the given hypothesis there exists a sequence $\{z_p\} \subset D$, $z_p \to 0$, and increasing sequences of natural numbers $\{n(p)\}$, $\{m(p)\}$ with $n(p) \neq m(p)$, such that $s_{m(p)}(z_p) \to w_1$ and $s_{n(p)}(z_p) \to w_2$ with $w_1 \neq w_2$. The assertion is clearly true if $s_n(0)$ diverges.

Suppose then that $s_n(0)$ converges, say $s_n(0) \to w_1$. Since $G(0; s)$ contains at least two points there exists $w_2 \neq w_1$, $z_p \to 0$, and $m(p)$ such that $s_{m(p)}(z_p) \to w_1$. Since the $s_n$ are continuous there exists a nested sequence of discs $R_n = \{z: |z| < \delta_n\}$, $\delta_n \to 0$, such that $z \in R_n$ implies $|s_n(0) - s_n(z)| < 1/n$. We may assume that each $z_p$ belongs to some $R_n$. For each $p$ let $R_{n(p)}$ be the smallest disc such that $z_p \in R_{n(p)}$. We may assume that the $n(p)$ are distinct (if not then we can choose a subsequence of $\{z_p\}$ for which this will be true). There are at most finitely many $p$ such that $z_p \in R_{m(p)}$. For if $z_p \in R_{m(p)}$ for infinitely many $p$ then by definition of $R_{m(p)}$ we would have $|s_{m(p)}(z_p) - s_{m(p)}(0)| < 1/m(p)$ and this implies that $w_1$ is a limit point of $s_{m(p)}(z_p)$ which is impossible. Thus $m(p) = n(p)$ for at most finitely many $p$. By discarding this finite number of indices we may assume that $m(p) \neq n(p)$, $p = 1, 2, \cdots$. We also have that $|s_{n(p)}(0) - s_{n(p)}(z_p)| < 1/n(p)$ and hence $s_{n(p)}(z_p) \to w_1$.

We now know that there exist sequences such that $s_{n(p)}(z_p) \to w_1$, $s_{m(p)}(z_p) \to w_2$ with $w_1, w_2 \in G(0; s)$, $w_1 \neq w_2$, and $n(p) \neq m(p)$. Given $w \in G(0; s)$ set $\alpha = (w - w_2)/(w_1 - w_2)$. Define the regular matrix $A$ by $a(k, n(k)) = \alpha$, $a(k, m(k)) = 1 - \alpha$, $a(k, j) = 0$ elsewhere. Then if $\{t_p\}$ is the $A$-transform of $s$, we have $t_p(z_p) = \alpha s_{n(p)}(z_p) + (1 - \alpha)s_{m(p)}(z_p)$ and hence $t_p(z_p) \to \alpha w_1 + (1 - \alpha)w_2 = w$. This completes the proof.

As a consequence of this last result we have the following

Theorem 7. Let $s = \{s_n(z)\}$ be a sequence of functions continuous and uniformly bounded in $D$, and let $s_n(z) \to f(z)$ in $D$. Suppose that $z_0 \in D$, that $K(z_0; s) \subseteq K(z_0; f)$, and that $K(z_0; s)$ contains more than one point. Then there exists a regular matrix $A$ such that $A$ induces the Gibbs phenomenon in $s$.

Proof. We need only choose $w \in K(z_0; f)$ and Theorem 6 assures us that there exists $A$ such that $w \in G(z_0; As) \subseteq K(z_0; As)$.

We can now answer the question posed at the beginning of this section. Izumi and Satô [4] have shown that there exists a function $f$
with a bounded discontinuity at \( x = 0 \) such that the Fourier series of \( f \) does not have the Gibbs phenomenon at \( x = 0 \). The Gibbs set \( G(0, s) \) of this Fourier series contains more than one point. Thus by Theorems 6 and 7 we have

**Theorem 8.** There exists a regular matrix \( A \) and a function \( f \) such that \( A \) induces the Gibbs phenomenon in the Fourier series of \( f \).

Finally then we may ask: What are necessary and sufficient conditions on a regular matrix \( A \) so that \( A \) does not induce the Gibbs phenomenon in Fourier series?

**References**


University of Cincinnati, Cincinnati, Ohio 45221