ON REGULAR MATRICES THAT INDUCE THE GIBBS PHENOMENON

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ABSTRACT. Let $s=\{s_n(z)\}$ be a sequence of complex valued functions defined in a subset D of the complex plane and suppose that $s_n(z)$ converges to $f(z)$ for $z\in D$. For $z_0\in\overline{D}$ let $K(z_0;\ s)$ and $K(z_0;f)$ be the cores of s and f respectively. We say that s does not have the Gibbs phenomenon at z_0 if $K(z_0;\ s)\subseteq K(z_0;f)$. The regular matrix A is said to induce the Gibbs phenomenon in s if $K(z_0;\ s)\subseteq K(z_0;f)$ but $K(z_0;\ As)\not\subseteq K(z_0;f)$. We characterize those regular matrices that induce the Gibbs phenomenon.

1. Let $s=\{s_n(z)\}$ be a sequence of complex valued functions defined in a subset D of the complex plane. Denote the closure of D by $\overline{D}$, and let $z_0\in\overline{D}$. Define the sets

$$G_n(z_0;s) = \{w: w = s_k(z),\ k \geqq n,\ z \in D,\ |z - z_0| < 1/n\}.$$

Then set $G(z_0;\ s)=\bigcap_{n=1}^{\infty}\overline{G}_n(z_0;\ s)$. It is easy to see that $G(z_0;\ s)$ is the set of complex numbers w of the form $w=\lim s_{n(k)}(z_k)$ where $n(1)<n(2)<\cdots$ and $z_k\to z_0$. When s is the sequence of partial sums of a real valued Fourier series the set $G(z_0;\ s)$ is called the Gibbs set of s at z_0. We will use the same terminology for $G(z_0;\ s)$ when s is any sequence. In the case of Fourier series $G(z_0;\ s)$ is connected as a consequence of the Riemann-Lebesgue Theorem. We will show that $G(z_0;\ s)$ is connected in the plane under certain conditions on s.

Let the closed convex hull of $G_n(z_0;\ s)$ be denoted by conv $G_n(z_0;\ s)$, and set $K(z_0;\ s)=\bigcap_{n=1}^{\infty}$ conv $G_n(z_0;\ s)$. The closed convex set $K(z_0;\ s)$ is called the core of s at z_0. The Gibbs set $G(z_0;f)$ and core $K(z_0;f)$ of a function $f(z)$ are the Gibbs set and core of the sequence $f_n(z)\equiv f(z)$. $G(z_0;f)$ is simply the cluster set of f at z_0. Suppose that s converges to $f(z)$ in D. We will say that s does not exhibit the Gibbs phenomenon at $z_0\in\overline{D}$ if $K(z_0;\ s)\subseteq K(z_0;\ f)$. This reduces to the usual definition when s is the partial sum sequence of a Fourier series. In §3 we will use the core definition to investigate the effect of regular matrix transformation on the Gibbs phenomenon.

The concept of core is due to K. Knopp who defined it for sequences of complex numbers. There is a large body of literature concerning cores of sequences and matrix summability. An account of this is given in [2] along with an extensive bibliography.

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2. In this section we will suppose that D is a locally connected subset of the complex plane. We will consider sequences that satisfy the following condition.

(α) There exists $\{\zeta_k\} \subset D$, $\zeta_k \to z_0 \in \overline{D}$, such that

$$\lim (n \to \infty) \; [s_{n+1}(\zeta_k) - s_n(\zeta_k)] = 0 \quad \text{for } k = 1, 2, \cdots.$$

LEMMA. *Let $s = \{s_n(z)\}$ be a sequence of continuous functions satisfying (α). Let the nonempty sets B_1 and B_2 satisfy the following conditions.*

(i) $G(z_0; s) \subseteq \text{int } B_1 \cup \text{int } B_2$, *where* int B_i *is the interior of* B_i.

(ii) $G(z_0; s) \cap \text{int } B_i \neq \emptyset$, $i = 1, 2$.

(iii) *There exist $\delta > 0$ and N such that $s_n(z) \in B_1 \cup B_2$ whenever $z \in D$ with $|z - z_0| < \delta$ and $n > N$.*

Then $d(B_1, B_2) = \inf\{|w_1 - w_2| : w_i \in B_i\} = 0$.

PROOF. Suppose that $d(B_1, B_2) = r > 0$. If $s_n(w) \in B_1$ with $|z_0 - w| \leqq \delta$ then $s_n(z) \in B_1$ for every $z \in D$ with $|z - z_0| \leqq \delta$, since s_n is continuous and B_1, B_2 are separated. By condition (α) there exists $\zeta_i \in \{\zeta_k\}$ such that $|\zeta_i - z_0| < \delta$ and there exists N such that $n \geqq N$ implies $|s_n(\zeta_i) - s_{n-1}(\zeta_i)| < r/2$. Suppose that $s_N(\zeta_i) \in B_1$. Since int $B_2 \cap G(z_0; s) \neq \emptyset$ there is a first index $M > N$ such that $s_M(\zeta_i) \in B_2$. Then $s_{M-1}(\zeta_i) \in B_1$ and hence $|s_M(\zeta_i) - s_{M-1}(\zeta_i)| \geqq r$. This contradicts the previous inequality and the lemma is proved.

We will say that a set Q is connected in the extended complex plane if in any separation $Q = Q_1 \cup Q_2$, both Q_1 and Q_2 are unbounded.

THEOREM 1. *Let s be a sequence of continuous functions satisfying condition (α). Then $G(z_0; s)$ is connected in the extended plane. If s is uniformly bounded in D then $G(z_0; s)$ is connected.*

PROOF. We will prove the second statement. The proof of the first is similar and is omitted. Suppose that $G(z_0; s)$ is not connected. Then there exist nonempty compact sets Q_1 and Q_2 such that $G(z_0; s) = Q_1 \cup Q_2$ and $d(Q_1, Q_2) = r_1 > 0$. Further, there exist compact sets B_1 and B_2 such that int $B_i \supset Q_i$ and $0 < d(B_1, B_2) = r_2 < r_1$. But then B_1 and B_2 satisfy the conditions of the lemma, and thus $d(B_1, B_2) = 0$. This contradiction establishes the result.

If s is real valued it is not necessary to distinguish between the bounded and unbounded case. It is perhaps worthwhile to state this as

THEOREM 2. *Let s be a sequence of continuous real functions that satisfy (α). Then $G(z_0; s)$ is connected.*

The sequence $\{e^{in}\}$ shows that (α) is not necessary for connectedness. Examples of sequences that satisfy (α) are sequences converging in D, and partial sums of Fourier series.

3. Let $A=(a_{nk})$ be a regular summability matrix. We will write $t=As$ for the transform sequence $t_n(z)=\sum a_{nk}s_k(z)$. In this section we will consider the following situation. Suppose that $s=\{s_n(z)\}$ converges pointwise to $f(z)$ in D. Suppose further that at $z_0\in\overline{D}$, $K(z_0;\, s)\subseteq K(z_0;f)$, so that s does not have the Gibbs phenomenon at z. Is it possible for As to show the phenomenon at z_0 even though s does not? We will show that this is indeed possible, and when this occurs we will say that A induces the Gibbs phenomenon in s. We will use the notation $\|A\|=\limsup(n\to\infty)\sum_k |a_{nk}|$.

THEOREM 3. *The regular matrix A induces the Gibbs phenomenon in some uniformly bounded sequence if and only if $\|A\|>1$.*

PROOF. Since A is regular, $\|A\|\geqq 1$. Suppose that $\|A\|=1$, and that s is a uniformly bounded sequence converging to f with $K(z_0;\, s)\subseteq K(z_0;\, f)$. Since $\|A\|=1$ the Bounded Core Theorem [1] implies $K(z_0;\, As)\subseteq K(z_0;\, s)$. Hence $K(z_0;\, As)\subseteq K(z_0;f)$ and A does not induce the Gibbs phenomenon. To prove the theorem in the opposite direction we will show that if $\|A\|>1$ then it is possible to construct a sequence with the desired property. The proof is an adaptation of a proof due to W. Hurwitz [3]. If $\|A\|>1$ then there exist $\lambda>0$ and sequences of natural numbers $\{n(p)\}$, $\{N(p)\}$ such that for $p=1, 2, \cdots$:

$$\sum_{k=1}^{N(p)} |a_{n(p),k}|>1+4\lambda,\qquad \sum_{k=1}^{N(p)-1} |a_{n(p),k}|<\lambda,\qquad \sum_{k=N(p)+1}^{\infty} |a_{n(p),k}|<\lambda.$$

For the purpose of the construction we assume that the complement of $\{N(p)\}$ in the natural numbers has the cardinality of the natural numbers. This assumption does not result in any loss of generality. Let $f(x)=\exp i/x$, $x\neq 0$, $-1\leqq x\leqq 1$ and $f(0)=0$.

Then $K(0,\, f)$ is the closed unit disc. Now define the sequence $s=\{s_k(x)\}$ as follows. We will write $x_p=1/n(p)$.

$$\begin{aligned}
s_k(0) &= 0, \qquad k=1,2,\cdots,\\
s_k(x) &= \exp i/x \quad \text{if } x\neq 0 \text{ and } x\neq 1/N(p), \qquad k=1,2,\cdots,\\
s_k(x_p) &= \exp iN(p), \qquad k\leqq N(p-1),\\
&= (-1)^p \exp(-i \arg a_{n(p),k}), \qquad N(p-1)<k\leqq N(p),\\
&= \exp iN(p), \qquad k>N(p).
\end{aligned}$$

Then $s_n(x) \to f(x)$ and $K(0; s)$ is the closed unit disc. Hence s does not exhibit the Gibbs phenomenon. Now

$$\begin{aligned} t_{n(p)}(x_p) &= \sum_k a_{n(p),k} s_k(x_p) \\ &= \sum_{k=1}^{N(p-1)} a_{n(p),k} s_k(x_p) + \sum_{k=N(p-1)+1}^{N(p)} a_{n(p),k} s_k(x_p) \\ &\quad + \sum_{k=N(p)+1}^{\infty} a_{n(p),k} s_k(x_p) \\ &= \exp iN(p) \sum_{k=1}^{N(p-1)} a_{n(p),k} - (-1)^p \sum_{k=1}^{N(p-1)} |a_{n(p),k}| \\ &\quad + (-1)^p \sum_{k=1}^{N(p)} |a_{n(p),k}| + \exp iN(p) \sum_{k=N(p)+1}^{\infty} a_{n(p),k}. \end{aligned}$$

Taking real parts:

$$\begin{aligned} (-1)^p \operatorname{Re} t_{n(p)}(x_p) &= \sum_{k=1}^{N(p)} |a_{n(p),k}| - \sum_{k=1}^{N(p-1)} |a_{n(p),k}| \\ &\quad + (-1)^p \operatorname{Re} \exp iN(p) \sum_{k=1}^{N(p-1)} a_{n(p),k} \\ &\quad + (-1)^p \operatorname{Re} \exp iN(p) \sum_{k=N(p)+1}^{\infty} a_{n(p)\,k} \\ &\geqq \sum_{k=1}^{N(p)} |a_{n(p),k}| - 2 \sum_{k=1}^{N(p-1)} |a_{n(p)\,k}| \\ &\quad - \sum_{k=N(p)+1}^{\infty} |a_{n(p),k}| \\ &> 1 + 4\lambda - 2\lambda - \lambda = 1 + \lambda. \end{aligned}$$

Now $(-1)^p \operatorname{Re}\, t_{n(p)}(x_p) > 1 + \lambda$ implies that $K(z_0; As)$ must have points lying outside the unit circle. Hence $K(0; f)$ does not contain $K(0; As)$ and A induces the Gibbs phenomenon in s.

It is possible to prove a theorem similar to Theorem 3 that holds for sequences of bounded functions. We must however restrict ourselves to row-finite regular matrices. The following terminology will be used. The row-finite matrix A will be said to be essentially real and nonnegative if the elements of A are real and nonnegative except possibly for finitely many rows or columns. A preliminary result will be required. This next theorem is an extension of the unbounded

core theorem for sequences of complex numbers [2, p. 148]. The proof is similar to that given in the reference and is omitted.

THEOREM 4. *Let A be row-finite and regular. Then $K(z_0;\ As) \subseteq K(z_0;\ s)$ for every sequence of bounded functions if and only if A is essentially real and nonnegative.*

THEOREM 5. *Let A be row-finite and regular. Then A does not induce the Gibbs Phenomenon in any sequence of bounded functions if and only if A is essentially real and nonnegative.*

PROOF. Let $s_n(z) \to f(z)$ and $K(z_0;\ s) \subseteq K(z_0; f)$. If A is essentially real and nonnegative then $K(z_0;\ As) \subseteq K(z_0;\ f)$ by Theorem 4. Hence A does not induce the Gibbs phenomenon. To prove the converse suppose first that $a_{n(p),k(p)} < 0$ where $n(1) < n(2) < \cdots$ and $k(1) < k(2) < \cdots$. Let $f(x) = 1/x$ for $0 < x \leqq 1$ and $f(0) = 0$. Set $x_p = 1/n(p)$. If $k \notin \{k(p)\}$ set $s_k(x) = 1/x$ for $1/k \leqq x \leqq 1$ and $s_k(x) = 0$ for $0 \leqq x < 1/k$. Set $s_{k(p)}(x_p) = -1/a_{n(p),k(p)}$; $s_{k(p)}(x) = 0$ for $0 \leqq x < x_p$ and $s_{k(p)}(x) = 1/x$ for $x_p < x \leqq 1$. Then $s = \{s_k(x)\}$ is a sequence of bounded functions converging to f, and $K(0,\ s) = K(0, f) = [0,\ \infty)$. But $t_{n(p)}(x_p) = -1$ and hence $K(0,\ As)$ is not contained in $K(0, f)$ .Thus A induces the Gibbs phenomenon. On the other hand, suppose that Im $a_{n(p),k(p)} \neq 0$ for $p = 1,\ 2,\ \cdots$. Say that Im $a_{n(p),k(p)} < 0$. Let $f(x) = i/x$ for $0 < x \leqq 1$ and $f(0) = 0$. If $k \notin \{k(p)\}$ then set $s_k(x) = i/x$ for $1/k \leqq x \leqq 1$; $s_k(x) = 0$ for $0 \leqq x < 1/k$. Also set $s_{k(p)}(x_p) = i/\text{Im } a_{n(p),k(p)}$; $s_{k(p)}(x) = 0$ if $0 \leqq x < x_p$, and $s_{k(p)}(x) = i/x$ if $x_p < x \leqq 1$. Again $K(0,\ s) = K(0, f)$ each set consisting of the imaginary axis iy, $y \geqq 0$. But Re $t_{n(p)}(x_p) = -1$ and hence $K(0,\ As)$ is not contained in $K(0,\ f)$. This completes the proof.

4. It is of interest to ask to what extent Theorems 3 and 5 apply to Fourier series. Clearly if $\|A\| = 1$ then A cannot induce the Gibbs phenomenon in uniformly bounded Fourier series. Also if A is essentially real and nonnegative, then A cannot induce the phenomenon in any Fourier series. But when A does not satisfy either of the above conditions there does not necessarily exist a Fourier sequence in which A induces the Gibbs phenomenon. Thus we can ask the following question: Does there exist a regular matrix A and a Fourier series in which A induces the Gibbs phenomenon?

We will answer this question affirmatively by proving a more general result. Previously we have considered $G(z_0;\ s)$ with $z_0 \in \overline{D}$ where D is the domain of definition of s. In this section we will take $z_0 \in D$ so that each s_n is defined at z_0.

THEOREM 6. *Let $s=\{s_n(z)\}$ be a sequence of functions continuous and uniformly bounded in D. Let $z_0\in D$ and let $G(z_0;\, s)$ contain more than one point. Then for every point w there exists a regular matrix A such that $w\in G(z_0;\, As)$.*

PROOF. We may take $z_0=0$. Our proof is based on the assertion that under the given hypothesis there exists a sequence $\{z_p\}\subset D$, $z_p\to 0$, and increasing sequences of natural numbers $\{n(p)\}$, $\{m(p)\}$ with $n(p)\neq m(p)$, such that $s_{m(p)}(z_p)\to w_1$ and $s_{n(p)}(z_p)\to w_2$ with $w_1\neq w_2$. The assertion is clearly true if $s_n(0)$ diverges.

Suppose then that $s_n(0)$ converges, say $s_n(0)\to w_1$. Since $G(0;\, s)$ contains at least two points there exists $w_2\neq w_1$, $z_p\to 0$, and $m(p)$ such that $s_{m(p)}(z_p)\to w_2$. Since the s_n are continuous there exists a nested sequence of discs $R_n=\{z: |z|<\delta_n\}$, $\delta_n\to 0$, such that $z\in R_n$ implies $|s_n(0)-s_n(z)|<1/n$. We may assume that each z_p belongs to some R_n. For each p let $R_{n(p)}$ be the smallest disc such that $z_p\in R_{n(p)}$. We may assume that the $n(p)$ are distinct (if not then we can choose a subsequence of $\{z_p\}$ for which this will be true). There are at most finitely many p such that $z_p\in R_{m(p)}$. For if $z_p\in R_{m(p)}$ for infinitely many p then by definition of $R_{m(p)}$ we would have $|s_{m(p)}(z_p)-s_{m(p)}(0)|<1/m(p)$ and this implies that w_1 is a limit point of $s_{m(p)}(z_p)$ which is impossible. Thus $m(p)=n(p)$ for at most finitely many p. By discarding this finite number of indices we may assume that $m(p)\neq n(p)$, $p=1,\, 2,\, \cdots$. We also have that $|s_{n(p)}(0)-s_{n(p)}(z_p)|<1/n(p)$ and hence $s_{n(p)}(z_p)\to w_1$.

We now know that there exist sequences such that $s_{n(p)}(z_p)\to w_1$, $s_{m(p)}(z_p)\to w_2$ with $w_1,\, w_2\in G(0;\, s)$, $w_1\neq w_2$, and $n(p)\neq m(p)$. Given $w\notin G(0;\, s)$ set $\alpha=(w-w_2)/(w_1-w_2)$. Define the regular matrix A by $a(k,\, n(k))=\alpha$, $a(k,\, m(k))=1-\alpha$, $a(k,\, j)=0$ elsewhere. Then if $\{t_p\}$ is the A-transform of s, we have $t_p(z_p)=\alpha s_{n(p)}(z_p)+(1-\alpha)s_{m(p)}(z_p)$ and hence $t_p(z_p)\to\alpha w_1+(1-\alpha)w_2=w$. This completes the proof.

As a consequence of this last result we have the following

THEOREM 7. *Let $s=\{s_n(z)\}$ be a sequence of functions continuous and uniformly bounded in D, and let $s_n(z)\to f(z)$ in D. Suppose that $z_0\in D$, that $K(z_0;\, s)\subseteq K(z_0;\, f)$, and that $K(z_0;\, s)$ contains more than one point. Then there exists a regular matrix A such that A induces the Gibbs phenomenon in s.*

PROOF. We need only choose $w\notin K(z_0;\, f)$ and Theorem 6 assures us that there exists A such that $w\in G(z_0;\, As)\subseteq K(z_0;\, As)$.

We can now answer the question posed at the beginning of this section. Izumi and Satô [4] have shown that there exists a function f

with a bounded discontinuity at $x=0$ such that the Fourier series of f does not have the Gibbs phenomenon at $x=0$. The Gibbs set $G(0, s)$ of this Fourier series contains more than one point. Thus by Theorems 6 and 7 we have

THEOREM 8. *There exists a regular matrix A and a function f such that A induces the Gibbs phenomenon in the Fourier series of f.*

Finally then we may ask: What are necessary and sufficient conditions on a regular matrix A so that A does not induce the Gibbs phenomenon in Fourier series?

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