TENSOR PRODUCTS OF BANACH ALGEBRAS. II

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0. In [1], [2] there are descriptions of an alleged bijection $\mathfrak{M}_3 \leftrightarrow \mathfrak{M}_1 \times \mathfrak{M}_2$, where $\mathfrak{M}_i$ is the set of regular maximal ideals of the Banach algebra A_i, $i=1, 2, 3$, and where $A_3 = A_1 \otimes_\gamma A_2$ is the greatest cross-norm tensor product of A_1 and A_2. The given constructions for the bijection are valid if both A_1 and A_2 are commutative. Otherwise, the constructions are in error (e.g., [1, p. 301, line 12]; [2, p. 753, line 4 and p. 755, Case 2]). The author knows of no counterexamples to the conclusions in [1], [2], [3].

The purpose of this note is to offer (correct) proofs of modified results in more general contexts and to apologize for the regrettable oversights. New research of K. Laursen, whose work (in prepublished form) the author had the privilege to examine, is based on the existence of the bijection mentioned above. Exposure to the new research led to the author's discovery of his earlier errors.

1. Let A_1, A_2 be (not necessarily commutative) Banach algebras with identities e_1 and e_2. Let $A_3 = A_1 \otimes_\gamma A_2$ (with identity $e_1 \otimes e_2$). For $i=1, 2, 3$ let $\mathfrak{L}_i$ denote the set of proper closed left ideals in A_i. We define two maps:

$$T: \mathfrak{L}_1 \times \mathfrak{L}_2 \to \mathfrak{L}_3, \qquad S: \mathfrak{L}_3 \to \mathfrak{L}_1 \times \mathfrak{L}_2$$

as follows:

ad T: If $L_i \in \mathfrak{L}_i$, $i = 1, 2$, define the linear map

$$R: A_3 \ni a_3 = \sum_n a_{1n} \otimes a_{2n} \to \sum_n (a_{1n}/L_1) \otimes (a_{2n}/L_2) \in (A_1/L_1) \otimes_\gamma (A_2/L_2).$$

Then $\ker(R) \equiv T(L_1, L_2) \in \mathfrak{L}_3$.

ad S: If $L_3 \in \mathfrak{L}_3$ define the linear maps

$$G_1: A_1 \ni a_1 \to (a_1 \otimes e_2)/L_3 \in A_3/L_3,$$
$$G_2: A_2 \ni a_2 \to (e_1 \otimes a_2)/L_3 \in A_3/L_3.$$

Then $\ker(G_i) \equiv L_i \in \mathfrak{L}_i$, $i=1, 2$. We set $S(L_3) = (L_1, L_2)$. (Alternatively: $S(L_3) = (L_3 \cap (A_1 \otimes e_2),\ L_3 \cap (e_1 \otimes A_2))$.)

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LEMMA 1. *Let* $L_3 \subset L_3'$, $S(L_3) = (L_1, L_2)$, $S(L_3') = (L_1', L_2')$. *Then* $L_i \subset L_i'$, $i = 1, 2$.

PROOF. Reference to the alternative definition of S leads directly to the result.

LEMMA 2. *If* $L_1 \subset L_1'$, $L_2 \subset L_2'$, $T(L_1, L_2) = L_3$, $T(L_1', L_2') = L_3'$, *then* $L_3 \subset L_3'$. *If* $L_1 \subsetneq L_1'$ *or* $L_2 \subsetneq L_2'$ *then* $L_3 \subsetneq L_3'$.

PROOF. Let $E_i: A_i \to A_i/L_i$, $E_i': A_i \to A_i/L_i'$ be the canonical maps, $i = 1, 2$. We may define a third map $F_i: A_i/L_i \to A_i/L_i'$, $i = 1, 2$, as follows. If $y \in A_i/L_i$, let $E_i(x) = y$ and let $F_i(y) = E_i'(x)$. If $E_i(\bar{x}) = y$ then $\bar{x} - x \in L_i \subset L_i'$ whence $E_i'(\bar{x}) = E_i(x)$ and thus $F_i(y)$ is well defined. Furthermore, if $z = E_i'(x)$ then $F_i E_i(x) = z$ and thus the following diagram

$$\begin{tikzcd} A_i \arrow[r, "E_i"] \arrow[dr, "E_i'"'] & A_i/L_i \arrow[d, "F_i"] \\ & A_i/L_i' \end{tikzcd}$$

$i = 1, 2$, is commutative. If $R': A_3 \to (A_1/L_1') \otimes_\gamma (A_2/L_2')$, then

$$\begin{aligned} R'\left(\sum_n a_{1n} \otimes a_{2n}\right) &= \sum_n (a_{1n}/L_1') \otimes (a_{2n}/L_2') \\ &= \sum_n E_1'(a_{1n}) \otimes E_2'(a_{2n}) \\ &= (F_1 \otimes F_2) \sum_n E_1(a_{1n}) \otimes E_2(a_{2n}) \\ &= F_1 \otimes F_2 \circ R\left(\sum_n a_{1n} \otimes a_{2n}\right). \end{aligned}$$

Thus $\ker(R) = T(L_1, L_2) \subset \ker(R') = T(L_1', L_2')$. If either $L_1 \subsetneq L_1'$ or $L_2 \subsetneq L_2'$ then correspondingly one of $\ker(F_1)$, $\ker(F_2)$ is nontrivial. Thus, e.g., if $0 \neq E_1(a_1) \in \ker(F_1)$, $E_2(a_2) \neq 0$, we find $a_1 \otimes a_2 \in \ker(R') \setminus \ker(R)$.

We shall consider together with $\mathfrak{L}_i$ the sets $\mathfrak{M}_{li}$, $\mathfrak{I}_i$, $\mathfrak{M}_i$ consisting respectively of maximal left ideals, closed ideals and maximal ideals in A_i, $i = 1, 2, 3$.

LEMMA 3. (a) $T(\mathfrak{I}_1 \times \mathfrak{I}_2) \subset \mathfrak{I}_3$;

(b) $S(\mathfrak{I}_3) \subset \mathfrak{I}_1 \times \mathfrak{I}_2$;

(c) *if* (J_1, J_2) *belongs to any of* $\mathfrak{L}_1 \times \mathfrak{L}_2$, $\mathfrak{M}_{l1} \times \mathfrak{M}_{l2}$, $\mathfrak{I}_1 \times \mathfrak{I}_2$ *or* $\mathfrak{M}_1 \times \mathfrak{M}_2$ *and if* $ST(J_1, J_2) = (J_1', J_2')$, *then* $J_i' \supset J_i$, $i = 1, 2$.

PROOF. The assertions (a) and (b) follow immediately from the relevant definitions. For (c) we note that if $a_1 \in J_1$, then $a_1/J_1 = 0$ whence $a_1 \otimes e_2 \in T(J_1, J_2) \equiv J_3$ whence $a_1 \in J_1'$. Thus $J_1 \subset J_1'$ and by a similar argument $J_2 \subset J_2'$.

COROLLARY. *On* $\mathfrak{M}_{I1} \times \mathfrak{M}_{I2}$ *and on* $\mathfrak{M}_1 \times \mathfrak{M}_2$ *the map* $ST = $ *identity.*

For each $I_i \in \mathfrak{J}_i$ let $\tilde{I}_i$ be some arbitrary but fixed element of $\mathfrak{M}_i$ and assume $\tilde{I}_i \supset I_i$. Define mappings $\tilde{S}$ and $\tilde{T}$ as follows:

$$\tilde{S}: \mathfrak{J}_3 \ni I_3 \to \tilde{S}(I_3) \equiv (\tilde{I}_1, \tilde{I}_2)$$

where $S(I_3) = (I_1, I_2)$.

$$\tilde{T}: \mathfrak{J}_1 \times \mathfrak{J}_2 \ni (I_1, I_2) \to \tilde{T}(I_1, I_2) = \tilde{I}_3$$

where $T(I_1, I_2) = I_3$. Despite the arbitrariness in the definitions of $\tilde{S}$ and $\tilde{T}$ we have

THEOREM 1. *On* $\mathfrak{M}_1 \times \mathfrak{M}_2$, $\tilde{S}\tilde{T} = $ *identity. If hk (hull-kernel) topologies are used throughout then* $\tilde{S}$ *is a continuous bijection of* $\tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2)$ *on* $\mathfrak{M}_1 \times \mathfrak{M}_2$. *(Note that* $\tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2) \subset \mathfrak{M}_3$.*)*

PROOF. Since $ST = $ identity on $\mathfrak{M}_1 \times \mathfrak{M}_2$ we see S is a bijection of $T(\mathfrak{M}_1 \times \mathfrak{M}_2)$ onto $\mathfrak{M}_1 \times \mathfrak{M}_2$. The parallel argument here reads: $T(M_1, M_2) = I_3 \subset \tilde{I}_3 = \tilde{T}(M_1, M_2)$. By Lemma 1, S is inclusion preserving. If $S(I_3) = (M_1, M_2)$ and $S(\tilde{I}_3) = (I_1, I_2)$ then $I_3 \subset \tilde{I}_3$ implies $M_i \subset I_i$ and so $M_i = I_i$ since M_i is maximal, $i = 1, 2$. Thus $\tilde{S}\tilde{T}(M_1, M_2) = \tilde{S}(\tilde{I}_3) = (\tilde{I}_1, \tilde{I}_2) = (M_1, M_2)$. Hence $\tilde{S}\tilde{T} = $ identity on $\mathfrak{M}_1 \times \mathfrak{M}_2$ and hence $\tilde{T}$ is injective; consequently $\tilde{S}$, on $\tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2)$, is bijective.

If F_1 is closed in $\mathfrak{M}_1$ then we show $\tilde{T}(F_1 \times \mathfrak{M}_2)$ is closed in $\tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2)$. A similar proof shows that if F_2 is closed in $\mathfrak{M}_2$ then $\tilde{T}(\mathfrak{M}_1 \times F_2)$ is closed in $\tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2)$. Since $\tilde{T}$ is injective and since any closed set in $\mathfrak{M}_1 \times \mathfrak{M}_2$ is of the form

$$\bigcap_\gamma [(F_1^\gamma \times \mathfrak{M}_2 \cup \mathfrak{M}_1 \times F_2^\gamma)]$$

where F_i^γ is closed in $\mathfrak{M}_i$, $i = 1, 2$, we see that

$$\tilde{T}\left(\bigcap_\gamma [(F_1^\gamma \times \mathfrak{M}_2) \cup (\mathfrak{M}_1 \times F_2^\gamma)]\right) = \bigcap_\gamma [\tilde{T}(F_1^\gamma \times \mathfrak{M}_2) \cup \tilde{T}(\mathfrak{M}_1 \times F_2^\gamma)]$$

is closed. Furthermore $\tilde{T}: \mathfrak{M}_2 \times \mathfrak{M}_1 \leftrightarrow \tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2)$ is closed, $\tilde{S}: \tilde{T}(\mathfrak{M}_1 \times \mathfrak{M}_2) \leftrightarrow \mathfrak{M}_1 \times \mathfrak{M}_2$ is $\tilde{T}^{-1}$ and thus $\tilde{S}$ is continuous. (This proof is essentially that given in [2]. There a counterexample shows $\tilde{T}$ need not be continuous.)

Thus let $M_3^0 = \tilde{T}(M_1^0, M_2^0) \supset k[\tilde{T}(F_1 \times \mathfrak{M}_2)]$. A contradiction will be obtained by constructing a single element $a_1^0 \otimes a_2^0 \in \tilde{T}(M_1, M_2) \backslash M_3^0$ for all $(M_1, M_2) \in F_1 \times \mathfrak{M}_2$. Thus $a_1^0 \otimes a_2^0 \in k[\tilde{T}(F_1 \times \mathfrak{M}_2)]$.

If $M_3^0 \notin \tilde{T}(F_1 \times \mathfrak{M}_2)$ then $M_1^0 \notin F_1$ and thus $M_1^0 \not\supset k(F_1)$. Let $a_1^0 \in k(F_1) \backslash M_1^0$. Then $a_1^0/M_1^0 \neq 0$. On the other hand, for $a_2 \in A_2$ consider $(a_1^0 \otimes a_2)/M_3$ where $M_3 = \tilde{T}(M_1, M_2)$, $(M_1, M_2) \in F_1 \times \mathfrak{M}_2$. Then since $a_1^0 \in k(F_1) \subset M_1$ we see $(a_1^0/M_1) \otimes (a_2/M_2) = 0$, i.e., $a_1^0 \otimes a_2 \in M_3$. Since (M_1, M_2) are arbitrary in $F_1 \times \mathfrak{M}_2$ we conclude $a_1^0 \otimes a_2 \in k[\tilde{T}(F_1 \times \mathfrak{M}_2)] \subset M_3^0$. Hence $(a_1^0 \otimes a_2)/M_3^0 = 0$. Now choose $a_2^0 \notin M_2^0$ (whence $a_2^0/M_2^0 \neq 0$). Then $(a_1^0/M_1^0) \otimes (a_2^0/M_2^0) \neq 0$ and so $a_1^0 \otimes a_2^0 \notin M_3^0 = \tilde{T}(M_1^0, M_2^0)$, a contradiction.

The argument in [**3**, p. 538, line 3 through p. 539, line 7] depends on the bijection $\mathfrak{M}_1 \times \mathfrak{M}_2 \leftrightarrow \mathfrak{M}_3$ alleged and discussed in [**2**]. Although the author knows of no instance where the bijection is absent, he is unaware of correct proofs. In what follows λ is the "least cross-norm whose associate is a cross-norm" [**4**, pp. 30–36] and τ is the extension of the identity map from the algebraic tensor product to the completions: $A_1 \otimes_\gamma A_2 \equiv A_3 \rightarrow A_1 \otimes_\lambda A_2$.

THEOREM 2. *Let A_1 and A_2 be strongly semisimple ($\bigcap_{\mathfrak{M}_i} M_i = \{0\}$, $i = 1, 2$) and assume that the map $\tau: A_1 \otimes_\gamma A_2 \equiv A_3 \rightarrow A_1 \otimes_\lambda A_2$ is* 1-1. *Furthermore assume that spectral synthesis holds for each element of $T(\mathfrak{M}_1 \times \mathfrak{M}_2)$, i.e., that $kh(T(M_1, M_2)) = T(M_1, M_2)$ for all (M_1, M_2) in $\mathfrak{M}_1 \times \mathfrak{M}_2$. Then $\bigcap \{M_3 : M_3 \in \bigcup_{\mathfrak{M}_1 \times \mathfrak{M}_2} h(T(M_1, M_2))\} \equiv P = \{0\}$ and in particular A_3 is strongly semisimple since $\bigcap_{\mathfrak{M}_3} M_3 \subset P = \{0\}$.*

PROOF. Let $a_3 \in \bigcap \{M_3 : M_3 \in \bigcup_{\mathfrak{M}_1 \times \mathfrak{M}_2} h(T(M_1, M_2))\}$. Then in particular $a_3 \in kh(T(M_1, M_2))$ for all $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$. By hypothesis, $a_3 \in T(M_1, M_2)$ for all $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$. Hence, if $a_3 = \sum_n a_{1n} \otimes a_{2n}$ then

$$\sum_n (a_{1n}/M_1) \otimes (a_{2n}/M_2) = 0$$

for all $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$. If $a_3 \neq 0$ then $\tau(a_3) \neq 0$ and thus $0 < \lambda(a_3) = \sup\{|\sum_n f_1(a_{1n}) f_2(a_{2n})| : (f_1, f_2) \in A_1^* \times A_2^*, |f_1| = |f_2| = 1\}$. Thus for some $(f_1, f_2) \in A_1^* \times A_2^*$ there obtains

$$\sum_n f_1(a_{1n}) f_2(a_{2n}) \neq 0.$$

We find in turn:

$$\sum_n f_1(a_{1n}) a_{2n} \neq 0;$$

for some M_2^0

$$\sum_n f_1(a_{1n})a_{2n}/M_2^0 \neq 0$$

since A_2 is semisimple;

$$\sum_n f_1(a_{1n})\psi_2(a_{2n}/M_2^0) \neq 0$$

for some $\psi_2 \in (A_2/M_2^0)^*$;

$$\sum_n \psi_2(a_{2n}/M_2^0)a_{1n} \neq 0;$$

for some M_1^0

$$\sum_n \psi_2(a_{2n}/M_2^0)(a_{1n}/M_1^0) \neq 0$$

since A_1 is semisimple. Thus for some $\psi_1 \in (A_1/M_1^0)^*$

$$\sum_n \psi_1(a_{1n}/M_1^0)\psi_2(a_{2n}/M_2^0) \neq 0.$$

Hence $0 < \lambda(\sum_n (a_{1n}/M_1^0) \otimes (a_{2n}/M_2^0)) \leqq \gamma(\sum_n (a_{1n}/M_1^0) \otimes (a_{2n}/M_2^0))$, a contradiction of $\sum_n (a_{1n}/M_1) \otimes (a_{2n}/M_2) = 0$ for all $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$.

The appearance of spectral synthesis in the above is of interest. Indeed, the basic and, as far as the author knows, unresolved question about the semisimplicity of $A_1 \otimes_\gamma A_2$ when A_1 and A_2 are semisimple (even commutative) is equivalent to spectral synthesis for the (reducing) ideal I of $F_\gamma(A_1, A_2)$ [**3**, pp. 526, 540]. Of course, semisimplicity of any Banach algebra A is the statement that spectral synthesis holds for the zero ideal $\{0\}$.

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