ON THE SOLUTION OF LINEAR FUNCTIONAL EQUATIONS BY AVERAGING ITERATION

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Curtis Outlaw and C. W. Groetsch [4] have recently shown that if \( T \) is an asymptotically convergent continuous linear self-mapping of a Banach space \( E \), and if \( f \) is in the range of \( I - T \), and \( 0 < \lambda < 1 \), and \( V_\lambda = \lambda I + (1 - \lambda)(T + f) \), then for each \( x \in E \) the sequence \( \{ V_\lambda^n x \} \) converges to a solution \( u \) of the equation \( u - Tu = f \). Since under these same hypotheses Browder and Petryshyn [1] showed that the sequence \( \{(T + f)^n x\} \) also converges to a solution \( u \) of \( u - Tu = f \), the Outlaw-Groetsch theorem essentially says that the averaged iteration \( x_{n+1} = V_\lambda x_n = \lambda x_n + (1 - \lambda)(Tx_n + f) \) yields a conservative process. The purpose of the present paper is to establish some fairly general conditions under which \( \{ V_\lambda^n x \} \) will converge to a solution \( u \) of \( u - Tu = f \) (even when \( \{(T + f)^n x\} \) does not).

Suppose \( E \) is a Banach space, \( T: E \rightarrow E \) is a continuous linear operator, and \( f \in E \). For \( 0 < \lambda < 1 \) we define \( S_\lambda = \lambda I + (1 - \lambda)T \), \( V_\lambda = \lambda I + (1 - \lambda)(T + f) \), and \( A_\lambda = [a_{n,j}] \) where \( a_{11} = 1 \), \( a_{ij} = 0 \) for \( j > 1 \), and for \( n > 1 \), \( a_{nj} = \frac{(-1)^{n-j}}{(n-1)!} \lambda^{n-j} (1 - \lambda)^{j-1} \) for \( 1 \leq j \leq n \), and \( a_{nj} = 0 \) for \( j > n \). It is easily seen that \( A_\lambda \) is a lower-triangular, nonnegative, infinite matrix with each row-sum equal to one and each column-limit equal to zero. For \( n > 1 \) we have the real polynomial \( a_n^\lambda(t) \) defined by

\[
S_{\lambda}^{n-1}(t) = (\lambda + (1 - \lambda)t)^{n-1} = \sum_{j=1}^{n} a_{n,j} t^{j-1} = a_n^\lambda(t).
\]

So, defining \( A_n^\lambda = a_n^\lambda(T) \), we have \( S_{\lambda}^{n-1} = A_n^\lambda \), since \( T \) is a linear operator. Defining

\[
b_n^\lambda(t) = (1 - a_n^\lambda(t))/(1 - t), \quad B_n^\lambda = b_n^\lambda(T),
\]

we have, for \( n \geq 2 \), \( B_n^\lambda = (1 - \lambda)[I + S_\lambda + S_\lambda^2 + \cdots + S_\lambda^{n-2}] \), since \( I - T = (1 - \lambda)^{-1}(I - S_\lambda) \). Also, for \( n \geq 2 \), we have

\[
V_\lambda^{n-1} = [S_\lambda + (1 - \lambda)f]^{n-1} = S_\lambda^{n-1} + (1 - \lambda)[I + S_\lambda + \cdots + S_\lambda^{n-2}](f) = A_n^\lambda + B_n^\lambda f,
\]

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since $S_\lambda$ is linear. It now follows at once from Theorem 3 of [3] that if $T$ is asymptotically $A_\lambda$-convergent (i.e., $\{I, T, T^2, \ldots\}$ is an asymptotically convergent semigroup with $\{A_\lambda^n\}$ as a system of almost invariant integrals) then $\{B_\lambda^n\}$ forms a system of companion integrals for $\{A_\lambda^n\}$ with respect to $T$. Consequently, with the above observation that $V_\lambda^{n-1}x = A_\lambda x + B_\lambda f$ for all $x \in E$ and all $n \geq 2$, Theorem 4 of [3] specializes to yield the following result.

**Theorem 1.** Suppose $T$ is an asymptotically $A_\lambda$-convergent continuous linear operator on the Banach space $E$, where $0 < \lambda < 1$, and suppose $f \in E$. Then, the following are true:

(a) If $f$ is in the range of $I - T$, then for any $x \in E$ the sequence $\{V_\lambda^n x\}$ converges to a solution $u$ of the equation $u - Tu = f$.

(b) If, for some $x \in E$, $\{V_\lambda^n x\}$ has a subsequence $\{V_\lambda^m x\}$ which converges weakly to a point $y \in E$, then $y - Ty = f$ and $\{V_\lambda^m x\}$ converges to $y$.

(c) If, for some $x \in E$, $\{V_\lambda^n x\}$ is contained in a weakly compact subset of $E$, then $\{V_\lambda^n x\}$ converges to a solution of the equation $u - Tu = f$.

In order to apply Theorem 1, one has to know only that the continuous linear operator $T$ is asymptotically $A_\lambda$-convergent for some $\lambda, 0 < \lambda < 1$. In this direction we have the following result.

**Theorem 2.** Suppose $T$ is a continuous linear operator on a uniformly convex Banach space $E$, and suppose $\|T\| \leq 1$. Then for any $\lambda, 0 < \lambda < 1$, $T$ is asymptotically $A_\lambda$-convergent.

**Proof.** By Theorem 5 of [3] it suffices to show that

(a) $T$ is asymptotically $A_\lambda$-bounded,

(b) $T$ is asymptotically $A_\lambda$-regular, and

(c) $\{A_\lambda^n x\}$ clusters weakly for each $x \in E$.

Since $\|T\| \leq 1$ we have

$$\|A_\lambda^n\| \leq \sum_{j=1}^{n} a_{n_j} \|T\|^{j-1} \leq 1$$

for all $n$, so that (a) is true. To get (b) we first observe that

$$TA_\lambda^n - A_\lambda^n = S_\lambda^{n-1} T - S_\lambda^{n-1} = (1 - \lambda)^{-1} [S_\lambda^n - S_\lambda^{n-1}],$$

so that $T$ will be asymptotically $A_\lambda$-regular (as defined in [3]) if and only if $S_\lambda$ is an asymptotically regular operator in the sense of Browder and Petryshyn [2]. Since $T$ is nonexpansive ($\|Tx - Ty\| = \|T(x - y)\| \leq \|T\| \cdot \|x - y\| \leq \|x - y\|$) and has at least one fixed point (viz. 0, since $T$ is linear), and since $E$ is a uniformly convex space, Theorem 5 of [2] gives us that for any $\lambda, 0 < \lambda < 1$, $S_\lambda$ is an asymptotically regular
operator. Hence we get (b). Finally, since uniformly convex Banach spaces are reflexive, closed spheres in \( E \) are weakly compact. Since for any \( x \in E \) we have for all \( n \)
\[
\| A_n^\lambda x \| \leq \| A_n \| \cdot \| x \| \leq \| x \|,
\]
it follows that for any \( x \in E \) the sequence \( \{ A_n^\lambda x \} \) clusters weakly; and so we get (c). Q.E.D.

**Corollary.** Suppose \( T \) is a continuous linear operator on a uniformly convex Banach space \( E \), and suppose \( \| T \| \leq 1 \). Then for any \( \lambda, 0 < \lambda < 1 \), and for any \( x \in E \) the sequence \( \{ S_n^\lambda x \} \) converges (strongly) to a fixed point of \( T \).

**Proof.** By Theorem 2, \( T \) is asymptotically \( A^\lambda \)-convergent. Since \( T \) is linear, \( (I - T)(0) = 0 \). Hence part (a) of Theorem 1 can be applied, with \( f = 0 \). But for \( f = 0 \) we have \( V_\lambda = S_\lambda \), and, of course, solutions of \( u - Tu = 0 \) are fixed points of \( T \). Q.E.D.

**Remark 1.** Setting \( \lambda = 1/2 \) in the above corollary provides an affirmative answer to a conjecture of Outlaw and Groetsch [4, p. 431].

**Remark 2.** It is easily seen that there are continuous linear operators which satisfy the hypotheses of Theorem 2, but which are not asymptotically convergent operators, e.g., any rotation of a finite-dimensional Euclidean space about the origin, or, in \( l^2 \), the shift operator \( (x_1, x_2, \ldots) \to (0, x_1, x_2, \ldots) \).

**References**


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