SHORTER NOTES

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CURVATURE FORMS FOR 2-MANIFOLDS

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The Gauss Bonnet formula is a well-known necessary condition for a 2-form to be the curvature form of a Riemannian metric on a 2-manifold. It appears to be not so well known that it is also sufficient. Precisely,

**Theorem.** Let $M$ be a compact, connected, orientable 2-dimensional manifold. Let $\Omega$ be a 2-form on $M$. Then a necessary and sufficient condition for $\Omega$ to be the curvature form of a Riemannian metric on $M$ is that

$$(1) \quad \int_M \Omega = 2\pi \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$.

**Proof.** As we have remarked, the necessity of (1) is the Gauss Bonnet Theorem. For the sufficiency, let $g$ be any Riemannian metric on $M$. We shall show that $\Omega$ is in fact the curvature form of a Riemannian metric conformal to $g$ (that is, of the form $e^{2\lambda}g$ for some $C^\infty$ function $\lambda$ on $M$). Let $\Omega$ be the curvature form for $g$. Then

$$(2) \quad \int_M (\Omega - \bar{\Omega}) = 0.$$

Now (2) is precisely the statement that $\Omega - \bar{\Omega}$ is orthogonal to the harmonic 2-forms on $M$. Thus by the Hodge Theorem, $\Omega - \bar{\Omega}$ is in the image of the Laplacian. Since $\Omega - \bar{\Omega}$ is a 2-form on a 2-manifold, this means that there is a 2-form $\beta$ such that

$$\Omega - \bar{\Omega} = d \ast d \ast \beta.$$

Let $\lambda = \ast \beta$. That $\bar{\Omega}$ is then the curvature form of the metric $\bar{g} = e^{2\lambda}g$.

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follows from the classical formula for change of curvature under conformal change of metric. We may compute this simply as follows. Let \( \{\omega_1, \omega_2\} \) be a local oriented orthonormal coframe field on \( M \) for the metric \( g \). If we set \( \tilde{\omega}_i = e^{\lambda} \omega_i \), then \( \{\tilde{\omega}_1, \tilde{\omega}_2\} \) is a local oriented orthonormal coframe field for \( \tilde{g} \). Now \( \Omega = d\phi_{12} \) where the Riemannian connection form \( \phi_{12} \) is uniquely determined by the requirements that \( \phi_{12} = -\phi_{21} \), \( d\omega_1 = -\phi_{12} \wedge \omega_2 \), and \( d\omega_2 = -\phi_{21} \wedge \omega_1 \). We compute \( \tilde{\phi}_{12} \). Let \( d\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2 \). Then

\[
d\tilde{\omega}_1 = e^{\lambda}(d\lambda \wedge \omega_1 - \phi_{12} \wedge \omega_2) = - (\lambda_2 \omega_1 - \lambda_1 \omega_2 + \phi_{12}) \wedge \tilde{\omega}_2.
\]

Thus \( \tilde{\phi}_{12} = \lambda_2 \omega_1 - \lambda_1 \omega_2 + \phi_{12} = \phi_{12} - *d\lambda \). Thus the curvature of the metric \( \tilde{g} = e^{2\lambda} g \) is \( d\tilde{\phi}_{12} = d\phi_{12} - d* d\lambda = \Omega - d*d\lambda = \tilde{\Omega} \).