A CHARACTERIZATION OF HOLOMORPHIC SEMIGROUPS

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ABSTRACT. A necessary and sufficient condition is given for a one-parameter semigroup $\{U(t)\}$, $0 \leqq t < \infty$, of class C_0 on a Banach space to be holomorphic (of class $H(\Phi_1, \Phi_2)$ for some $\Phi_1 < 0 < \Phi_2$). The condition is expressed in terms of the spectral properties of $U(t) - \zeta$ for small $t > 0$ and for a complex number ζ with $|\zeta| \geqq 1$.

Let $\{U(t)\}_{0<t<\infty}$ be a semigroup of linear operators of class C_0 on a (complex) Banach space X (see Hille-Phillips [1, p. 321]). In this note we give a necessary and sufficient condition for $\{U(t)\}$ to be a holomorphic semigroup, by which we mean a semigroup belonging to the class $H(\Phi_1, \Phi_2)$ for some $\Phi_1 < 0 < \Phi_2$ [1, p. 325]. Roughly speaking, we prove that $\{U(t)\}$ is holomorphic if and only if there is a point on the unit circle $|\zeta| = 1$, different from 1, that belongs to the resolvent set $\rho(U(t))$ of $U(t)$ for sufficiently small t.

In this connection it will be noted that for any $\{U(t)\}$ of class C_0, each complex number ζ with $|\zeta| > 1$ belongs to $\rho(U(t))$ for sufficiently small t. In other words, the spectral radius $r(U(t))$ of $U(t)$ satisfies $\limsup_{t\to 0} r(U(t)) \leqq 1$. This follows immediately from the well-known inequality

$$\|U(t)\| \leqq Me^{\beta t}, \qquad 0 < t < \infty, \tag{1}$$

where M and β are constants.

Our result generalizes a recent result of Neuberger [2] to the effect that if $\limsup_{t\to 0} \|1 - U(t)\| < 2$ then $AU(t) \in B(X)$ for $t > 0$, where A denotes the infinitesimal generator of $\{U(t)\}$.

Precisely, our result is given by

THEOREM. *Let $\{U(t)\}$ be a semigroup of class C_0. The following three conditions are equivalent.*

(i) *$\{U(t)\}$ is a holomorphic semigroup.*

(ii) *For each complex number ζ with $|\zeta| \geqq 1$, $\zeta \neq 1$, there are $\delta, K > 0$ such that*

$$\zeta \in \rho(U(t)), \qquad \|(\zeta - U(t))^{-1}\| \leqq K \quad \text{for } 0 < t < \delta.$$

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(iii) *There are positive numbers* δ, K *and a complex number* ζ *with* $|\zeta| = 1$ *such that*

$$\|(\zeta - U(t))x\| \geqq \|x\|/K \quad \text{for } x \in X \text{ and } 0 < t < \delta.$$

COROLLARY. *If there are* δ, $\epsilon > 0$ *such that* $\|1 - U(t)\| \leqq 2 - \epsilon$ *for* $0 < t < \delta$, *then* $\{U(t)\}$ *is holomorphic.*

To deduce the corollary from the theorem, it suffices to note that (iii) is satisfied with $\zeta = -1$. In fact $\|(-1 - U(t))x\| \geqq 2\|x\| - \|(1 - U(t))x\| \geqq \epsilon\|x\|$.

To prove the theorem, we use the infinitesimal generator A of $U(t)$. Suppose $\{U(t)\}$ is holomorphic. Then $\rho(A)$ contains a sector $S = \{z \mid \arg(z - \beta) < \pi/2 + \omega\}$ with $\omega > 0$, and

$$\|(z - A)^{-1}\| \leqq M/|z - \beta|, \qquad z \in S. \tag{2}$$

$U(t)$ is given by

$$U(t) = \frac{1}{2\pi i}\int_C e^{tz}(z - A)^{-1}dz, \qquad t > 0, \tag{3}$$

where C is a curve running in S from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ with $\pi/2 < \theta < \pi/2 + \omega$. (These results are well known and may be found in [1], [3] or [4].)

Changing the variable of integration, we obtain

$$U(t) = \frac{1}{2\pi i}\int_C e^{z}(z - tA)^{-1}dz, \qquad t > 0, \tag{4}$$

where we may assume that C is independent of t as long as $0 < t < \delta$. Furthermore, the portion of C in the right half-plane can be made as small as we like if δ is sufficiently small. In particular, given any $\zeta \neq 1$ with $|\zeta| \geqq 1$, we can achieve that $e^z \neq \zeta$ for all z lying on C or to the left of C.

With such a choice of δ and C, we define

$$B(t) = \frac{1}{2\pi i}\int_C e^{z}(e^z - \zeta)^{-1}(z - tA)^{-1}dz, \qquad 0 < t < \delta. \tag{5}$$

The integral converges absolutely and defines an operator $B(t) \in B(X)$ (the set of all bounded linear operators on X to X). Since (2) implies $\|(z - tA)^{-1}\| \leqq M/|z - t\beta| \leqq M'/|z|$ for $z \in C$, we have

$$\|B(t)\| \leqq (M'/2\pi)\int_C |z^{-1}e^{z}(e^z - \zeta)^{-1}dz| = M'', \qquad 0 < t < \delta. \tag{6}$$

Now we apply the Dunford-Taylor operational calculus to (4), (5), noting that the functions e^z and $e^z(e^z-\zeta)^{-1}$ of z are holomorphic on and to the left of C and tend rapidly to zero as $\operatorname{Re} z \to -\infty$. Since the product of these two functions are equal to $e^z+\zeta e^z(e^z-\zeta)^{-1}$, we obtain $U(t)B(t)=B(t)U(t)=U(t)+\zeta B(t)$. Hence $(\zeta-U(t))(1-B(t))=\zeta=(1-B(t))(\zeta-U(t))$ and so

$$(\zeta - U(t))^{-1} = \zeta^{-1}(1 - B(t)), \qquad \|(\zeta - U(t))^{-1}\| \leqq |\zeta|^{-1}(1 + M'').$$

This proves that (i) implies (ii).

It is obvious that (ii) implies (iii).

To prove that (iii) implies (i), we use the identity

$$e^{-it\alpha}U(t)x - x = \int_0^t e^{-is\alpha}U(s)(A - i\alpha)x\,ds, \qquad x \in D(A),$$

where α is any real number. Using (1), we thus obtain

$$\begin{aligned} \|(U(t) - e^{it\alpha})x\| &\leqq M\beta^{-1}(e^{\beta t} - 1)\|(A - i\alpha)x\| \\ &\leqq Mt(1 - \beta t)^{-1}\|(A - i\alpha)x\| \end{aligned} \tag{7}$$

for $\beta t<1$.

Suppose (iii) is satisfied. Choose two positive numbers θ', θ'' such that $\zeta=e^{i\theta'}=e^{-i\theta''}$. Let $\alpha>\max(\theta'/\delta,\ \beta\theta')$ and set $t=\theta'/\alpha$. Then $0<t<\delta$ and $\beta t<1$. Since $e^{it\alpha}=e^{i\theta'}=\zeta$, we have $\|(U(t)-e^{it\alpha})x\| \geqq \|x\|/K$ by hypothesis. It follows from (7) that

$$\begin{aligned} \|(A - i\alpha)x\| &\geqq (1 - \beta t)\|x\|/MKt \\ &= (\alpha - \beta\theta')\|x\|/MK\theta', \qquad x \in D(A). \end{aligned} \tag{8}$$

(8) implies that if $\alpha>\beta\theta'$, $A-i\alpha$ is a semi-Fredholm operator with nullity 0 and minimum modulus $\geqq(\alpha-\beta\theta')/MK\theta'$ (see [4, p. 230]). Hence $A-i\alpha-\xi$ is also semi-Fredholm if $|\xi|<(\alpha-\beta\theta')/MK\theta'$ (see [4, p. 236]). But it is known that $i\alpha+\xi\in\rho(A)$ if $\xi>\beta$, so that $\operatorname{ind}(A-i\alpha-\xi)=0$. Since the index is constant on any component of the semi-Fredholm domain (see [4, p. 243]), we conclude that $\operatorname{ind}(A-i\alpha)=0$ or, equivalently, $i\alpha\in\rho(A)$, provided α is sufficiently large. Then (8) implies that $\|(A-i\alpha)^{-1}\|\leqq MK\theta'/(\alpha-\beta\theta')$ for sufficiently large α.

Similarly one proves that $-i\alpha\in\rho(A)$ for sufficiently large α, with $\|(A+i\alpha)^{-1}\|\leqq MK\theta''/(\alpha-\beta\theta'')$. As is well known (see [1], [3], or [4]), these inequalities show that $\{U(t)\}$ is holomorphic. Thus (iii) implies (i).

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