# MAXIMAL TWO-SIDED IDEALS IN TENSOR PRODUCTS OF BANACH ALGEBRAS

KJELD B. LAURSEN

In [1] and [2] a construction of a bijection $\mathfrak{M}_3 \leftrightarrow \mathfrak{M}_1 \times \mathfrak{M}_2$ is given, where $\mathfrak{M}_i$ is the set of maximal modular (two-sided) ideals in the Banach algebra $A_i$ $(i=1, 2, 3)$ and where $A_3 = A_1 \otimes_\gamma A_2$ is the greatest cross-norm tensor product of $A_1$ and $A_2$. In a recent correction [3] it is shown that there is indeed a closed 1-1 mapping $\mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3$ when hull-kernel topologies are used. However, it is an open question when this mapping is surjective. In this note we show that the mapping is onto when one of the Banach algebras $A_1$ and $A_2$ is commutative. Also we give a correct proof of a theorem in [5], the original proof depended on [2].

The methods employed are adaptions of those in [6].

Suppose $A_1$ and $A_2$ are Banach algebras and suppose $A_1$ is commutative. Let $\mathfrak{M}_i$ be the set of maximal modular (two-sided) ideals. Each $h \in \mathfrak{M}_1$ is a continuous $\mathbf{C}$-valued homomorphism and induces a homomorphism

$$\phi_h : A_3 \to A_2$$

defined by

$$t = \sum_{i=1}^{\infty} a_{1i} \otimes a_{2i} \in A_3 \Rightarrow \phi_h(t) = \sum_{i=1}^{\infty} h(a_{1i}) a_{2i} \in A_2.$$

THEOREM 1. *If $A_1$, $A_2$, $A_3 = A_1 \otimes A_2$ are Banach algebras with spaces $\mathfrak{M}_i$ of maximal modular ideals and if $A_1$ is commutative then there is a bijection*

$$\tilde{T} : \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3.$$

*This bijection is given by the following:*

*$M_3 \subseteq A_3$ is an element of $\mathfrak{M}_3$ if and only if there is a continuous homomorphism $h \in \mathfrak{M}_1$ and a maximal modular ideal $M_2 \in \mathfrak{M}_2$ such that*

$$M_3 = \phi_h^{-1}(M_2) \equiv \tilde{T}(h, M_2).$$

*When the algebras have identities the mapping $\tilde{T}$ is identical with the one defined in [3] and, consequently, is closed with respect to the hull-kernel topologies.*

Received by the editors January 29, 1969.

PROOF. Sufficiency. Suppose $h\in\mathfrak{M}_1$, $M_2\in\mathfrak{M}_2$ with identities modulo the ideal $u_1$ and $u_2$, respectively, and suppose $M_3=\phi_h^{-1}(M_2)$. Clearly, $M_3$ is a modular ideal with $u_1\otimes u_2$ as an identity modulo $M_3$. Suppose $M_3'\supsetneqq M_3$. From the definition of $M_3$ we get $\phi_h(M_3')\supsetneqq M_2$, i.e. $\phi_h(M_3')=A_2$, since $M_2$ is maximal. Therefore, there is $t\in M_3'$ so that $\phi_h(t)=u_2$. Now, $\phi_h(t-u_1\otimes u_2)=\phi_h(t)-h(u_1)u_2=u_2-u_2=0$ so $t-u_1\otimes u_2 \in M_3\subseteq M_3'$; hence $u_1\otimes u_2\in M_3'$ and $M_3'=A_3$. It follows that $M_3$ is maximal.

To prove that every $M_3\in\mathfrak{M}_3$ is of the form $\phi_h^{-1}(M_2)$ for some $h\in\mathfrak{M}_1$, $M_2\in\mathfrak{M}_2$ suppose first that $A_1$ and $A_2$ have identities, both denoted by $e$. Let $M_3\in\mathfrak{M}_3$. Let $\pi:A_3\to A_3/M_3$ be the canonical mapping and let $\pi_i$ be the restriction of $\pi$ to $A_i$ via the identifications

$$A_1\leftrightarrow A_1\otimes e\subseteq A_3,\qquad A_2\leftrightarrow e\otimes A_2\subseteq A_3.$$

The kernel of $\pi_1$ is easily seen to be $A_1\cap M_3$ so in the diagram below provided us by the induced mapping theorem $U:A_1/A_1\cap M_3\to A_3/M_3$ is seen to be an isomorphism.

$$\begin{tikzcd} A_1 \arrow[rr, "\pi_1"] \arrow[dr, "\sigma"'] & & A_3/M_3 \\ & A_1/A_1\cap M_3 \arrow[ur, "U"'] & \end{tikzcd}$$

Since $A_1$ is commutative $\pi_1(A_1)$ is an abelian subalgebra of $A_3/M_3$. Since $\pi_1(A_1)$ and $\pi_2(A_2)$ commute elementwise, $\pi_1(A_1)\subseteq$ center$(\pi_1(A_1)\pi_2(A_2))$. Consequently, from $A_1\otimes A_2$ being dense in $A_1\otimes_\gamma A_2$ we infer that $\pi_1(A_1)\subseteq$center $A_3/M_3$. From the diagram we then conclude $U(A_1/A_1\cap M_3)=U(\sigma(A_1))=\pi_1(A_1)\subseteq$center$(A_3/M_3)$. Therefore $A_1/A_1\cap M_3\cong\mathbf{C}$ because $A_3/M_3$ is primitive [7, p. 61]. It follows that $A_1\cap M_3$ is a maximal ideal of $A_1$ and consequently is an element, $h$, of $\mathfrak{M}_1$.

Next we show that kernel $(\phi_h)\subseteq M_3$. But this follows immediately from the observation that

$$\pi=\pi_2\circ\phi_h$$

together with kernel $(\pi)=M_3$. ($\pi=\pi_2\circ\phi_h$ is seen to hold by noting that both sides of the equation are continuous and coincide on $A_1\otimes A_2$, i.e. everywhere.)

We now show that $M_3=\phi_h^{-1}(M_2)$ where $M_2\in\mathfrak{M}_2$. Since $\phi_h$ is surjective, $\phi_h(M_3)$ is an ideal in $A_2$. If $\phi_h(M_3)=A_2$ then there is $t\in M_3$ so that $\phi_h(t)=e\in A_2$ and hence $\phi_h(t-e\otimes e)=0$. Since kernel $(\phi_h)\subseteq M_3$, $e\otimes e\in M_3$, i.e. $M_3=A_3$. It follows that $\phi_h(M_3)$ is a proper ideal. Let

$M_2 \supseteq \phi_h(M_3)$ be a maximal ideal. We claim that $M_3 = \phi_h^{-1}(M_2)$. Since $M_3$ as well as $\phi_h^{-1}(M_2)$ are maximal (see the sufficiency part of this proof) it suffices to note that $M_3 \subseteq \phi_h^{-1}(M_2)$.

Before removing the assumption concerning the presence of identities we show that the mapping defined above is identical with that of [**3**].

Let $\mathfrak{F}_i$ be the set of closed (two-sided) ideals in $A_i$ $(i=1, 2, 3)$. Then Gelbaum defines the following mappings $S$ and $T$.

If $I_i \in \mathfrak{F}_i$ $(i=1, 2)$ then

$$R: A_3 \ni t = \sum a_{1i} \otimes a_{2i} \to \sum (a_{1i}/I_1) \otimes (a_{2i}/I_2) \\ \in A_1/I_1 \ \otimes_\gamma A_2/I_2$$

and $T(I_1, I_2) = \text{kernel } (R) \in \mathfrak{F}_3$.

If $I_3 \in \mathfrak{F}_3$ then

$$G_1: A_1 \ni a_1 \to a_1 \otimes e/I_3 \in A_3/I_3,$$
$$G_2: A_2 \ni a_2 \to e \otimes a_2/I_3 \in A_3/I_3;$$

kernel $(G_i) = I_i$, $i=1, 2$ and $S(I_3) = (I_1, I_2) \in \mathfrak{F}_1 \times \mathfrak{F}_2$. Note that $S(I_3) = (I_3 \cap A_1 \otimes e, I_3 \cap e \otimes A_2)$.

It is then shown that $ST$ is the identity on $\mathfrak{M}_1 \times \mathfrak{M}_2$ and that $S$ induces a mapping $\tilde{S}: \mathfrak{M}_3 \to \mathfrak{M}_1 \times \mathfrak{M}_2$, while $T$ induces $\tilde{T}: \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3$ with the property that $\tilde{S}\tilde{T}$ is the identity on $\mathfrak{M}_1 \times \mathfrak{M}_2$. Actually $\tilde{T}$ and $\tilde{S}$ are defined as choice functions; if $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$ then $T(M_1, M_2) \in \mathfrak{F}_3$. Take any $M_3 \in \mathfrak{M}_3$ such that $M_3 \supseteq T(M_1, M_2)$ and define $\tilde{T}(M_1, M_2) = M_3$. Similarly, if $M_3 \in \mathfrak{M}_3$ and $S(M_3) = (I_1, I_2) \in \mathfrak{F}_1 \times \mathfrak{F}_2$ pick $M_i \in \mathfrak{M}_i$ such that $I_i \subseteq M_i$ $(i=1, 2)$ and define $\tilde{S}(M_3) = (M_1, M_2)$. We shall show that in our situation $T(h, M_2) = \phi_h^{-1}(M_2)$ for every $(h, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$; this will show that $\tilde{T}$ is surjective and that $\tilde{S} = \tilde{T}^{-1}$. Let $(h, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$; then $T(h, M_2) = \text{kernel } (R)$, where

$$\begin{aligned} R: t = \sum a_{1i} \otimes a_{2i} &\to \sum a_{1i}/h \otimes a_{2i}/M_2 \\ &= \sum h(a_{1i}) a_{2i}/M_2 \\ &= \left(\sum h(a_{1i}) a_{2i}\right)/M_2 \\ &= \phi_h(t)/M_2, \end{aligned}$$

i.e. $t \in T(h, M_2)$ if and only if $\phi_h(t) \in M_2$ so

$$T(h, M_2) = \phi_h^{-1}(M_2).$$

It follows that $M_3 = T(h, M_2) \in \mathfrak{M}_3$. We have shown that Gelbaum's mapping $\tilde{T}$ is identical with that introduced here. Theorem 1 [3] gives us the result that $\tilde{T}$ is closed with respect to the hull-kernel topologies.

If $A_1$ and $A_2$ do not necessarily have identities, let $\tilde{A}_i$ $(i=1, 2)$ be the smallest algebra with an identity containing $A_i$ and let $\tilde{A}_3=\tilde{A}_1 \otimes_\gamma \tilde{A}_2$. Then $A_3$ can be viewed as an ideal in $\tilde{A}_3$ [2, Lemma 2]. What remains to be shown is that if $M_3 \in \mathfrak{M}_3$ then we can find $h \in \mathfrak{M}_1$ and $M_2 \in \mathfrak{M}_2$ such that $M_3=\phi_h^{-1}(M_2)$. Let $M_3 \in \mathfrak{M}_3$ and let $u$ be an identity mod $M_3$. Define

$$\tilde{M}_3 = \{t \in \tilde{A}_3, ut \in M_3 \text{ and } tu \in M_3\}.$$

Then the following holds:

(a) $M_3 \subseteq \tilde{M}_3$, and

(b) $u$ is an identity (in $\tilde{A}_3$) mod $\tilde{M}_3$.

(a) is clear, and (b) is proved in [2, Lemma 4].

Let $\tilde{M}_3'$ be a maximal modular ideal containing $\tilde{M}_3$. Since $\tilde{M}_3'$ is proper $u \notin \tilde{M}_3'$. By the part already proved we can find maximal modular ideals $\tilde{h} \subseteq \tilde{A}_1$ and $\tilde{M}_2 \subseteq \tilde{A}_2$ such that

$$\tilde{M}_3' = \overset{-1}{\phi_{\tilde{h}}}(M_2).$$

If $\tilde{h} \supseteq A_1$ or if $\tilde{M}_2 \supseteq A_2$ then $\phi_{\tilde{h}}(u)=0$, i.e. $u \in \tilde{M}_3'$ which is a contradiction. It follows that $\tilde{h} \cap A_1 \equiv h \in \mathfrak{M}_1$ and $\tilde{M}_2 \cap A_2 \equiv M_2 \in \mathfrak{M}_2$. Left to show is that

$$M_3 = \overset{-1}{\phi_h}(M_2).$$

Both sides of this expression are maximal ideals so it suffices to show

$$M_3 \subseteq \overset{-1}{\phi_h}(M_2);$$

but

$$\begin{aligned} t \in M_3 &\Rightarrow t \in A_3 \cap \tilde{M}_3' \\ &\Rightarrow \phi_{\tilde{h}}(t) \in \tilde{M}_2 \cap A_2 = M_2. \end{aligned}$$

Also $\phi_{\tilde{h}}(t)=\phi_h(t)$, since $h=\tilde{h} \cap A_1$, so

$$\phi_h(M_3) \subseteq M_2 \quad \text{i.e. } M_3 \subseteq \overset{-1}{\phi_h}(M_2).$$

This finishes the proof.

A Banach algebra is strongly semisimple if the intersection of all maximal modular two-sided ideals is $\{0\}$.

Let $\lambda$ be the "least cross-norm" [8] and let $\tau: A_1 \otimes_\gamma A_2 \rightarrow A_1 \otimes_\lambda A_2$ be the canonical mapping (the extension to all of $A_1 \otimes_\gamma A_2$ of the identity on the algebraic tensor product $A_1 \otimes A_2$).

THEOREM 2 [5]. *Suppose $A_1$ and $A_2$ are strongly semisimple Banach algebras and suppose $A_1$ is commutative. Then $A_1 \otimes_\gamma A_2$ is strongly semisimple if and only if*

$$\tau: A_1 \otimes_\gamma A_2 \to A_1 \otimes_\lambda A_2$$

*is* 1-1.

PROOF. Suppose $\tau$ is 1-1. Since $A_1$ is commutative, Theorem 1 tells us that Theorem 2 [3] applies so $A_3 = A_1 \otimes_\gamma A_2$ is strongly semisimple. Suppose conversely that $A_1 \otimes_\gamma A_2$ is strongly semisimple. Then if $t \in A_3$ and $t \neq 0$ we can find $M_3 \in \mathfrak{M}_3$ so that $t \notin M_3 = \phi_h^{-1}(M_2)$ for some $(h, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$ (by Theorem 1). Consequently, $\phi_h(t) \notin M_2$ and in particular $\phi_h(t) \neq 0$ so there is $a_2^* \in A_2^*$ (the dual space of $A_2$) for which $a_2^*(\phi_h(t)) \neq 0$. Since $\lambda(\tau(t)) = \sup |a_1^* \otimes a_2^*(t)|$ where the supremum is taken when $a_i^*$ ranges over the unit ball of $A_i^*$ $(i = 1, 2)$ and since $a_2^*(\phi_h(t)) = h \otimes a_2^*(t)$, as an easy computation shows, we conclude that $\lambda(\tau(t)) \neq 0$ and hence that $\tau$ is 1-1.

REMARK. It is easy to see that if $A_1$ is commutative and $A_1 \otimes_\gamma A_2$ is strongly semisimple then $A_1$ and $A_2$ are strongly semisimple: $a_2 \in \bigcap_{M_2 \in \mathfrak{M}_2} M_2$ if and only if $\phi_h^{-1}(a_2) = 0$ for every $h \in \mathfrak{M}_1$ by Theorem 1 and the strong semisimplicity of $A_1 \otimes_\gamma A_2$. Since $\phi_h(a_1 \otimes a_2) = h_1(a_1)a_2$ it follows that $a_1 \otimes a_2 = 0$ for any $a_1 \in A_1$, i.e. $a_2 = 0$. Similarly, $a_1 \in \bigcap_{h \in \mathfrak{M}_1} h$ implies that $\phi_h(a_1 \otimes a_2) = 0$ for every $h \in \mathfrak{M}_1$ and every $a_2 \in A_2$, i.e. $a_1 \otimes a_2 = 0$ by the strong semisimplicity of $A_1 \otimes_\gamma A_2$ so $a_1 = 0$.

COROLLARY 1. *Let $G$ be a locally compact abelian group, $A$ a Banach algebra and $L^1(G, A)$ the algebra of Bochner-integrable functions from $G$ to $A$ with convolution multiplication. Then $L^1(G, A)$ is strongly semisimple if and only if $A$ is strongly semisimple.*

PROOF. $L^1(G, A) = L^1(G) \otimes_\gamma A$ [4]. Moreover, $\tau: L^1(G) \otimes_\gamma A \to L^1(G) \otimes_\lambda A$ is 1-1 since $L^1(G)$ satisfies Grothendieck's condition of approximation [4]. The sufficiency then follows from Theorem 2, since $L^1(G)$ is known to be (strongly) semisimple [9]. The necessity is covered by the above remark.

COROLLARY 2 [9, THEOREM 1.7]. *Let $G$ be an abelian locally compact group and $H$ a compact group. Then $L^1(G \times H)$ is strongly semisimple.*

PROOF. $L^1(G \times H) = L^1(G) \otimes_\gamma L^1(H)$ [4] and $L^1(H)$ is strongly semisimple [9].

## References

1. B. R. Gelbaum, *Tensor products of Banach algebras*, Canad. J. Math. **11** (1959), 297–310. MR **21** #2922.

2. ———, *Note on the tensor product of Banach algebras*, Proc. Amer. Math. Soc. **12** (1961), 750–757. MR **24** #A1039.

3. ———, *Tensor products of Banach algebras*, Corrections of earlier work, 1968 (preprint).

4. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955), MR **17**, 763.

5. K. B. Laursen, *Tensor products of Banach-* $*$*-algebras*, Thesis, University of Minnesota, Minneapolis, Minn., 1967.

6. A. Lebow, *Maximal ideals in tensor products of Banach algebras*, Bull. Amer. Math. Soc. **74** (1968), 1020–1022. MR **38** #539.

7. C. Rickart, *General theory of Banach algebras*, The University Series in Higher Mathematics, Van Nostrand, Princeton, N. J., 1960. MR **22** #5903.

8. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, no. 26, Princeton Univ. Press, Princeton, N. J., 1950. MR **12**, 186.

9. I. E. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc. **61** (1947), 69–105. MR **8**, 438.

UNIVERSITY OF AARHUS, AARHUS, DENMARK