SOLUTIONS OF $f(x) = f(a) + (RL) \int_a^x (fH + fG)$ FOR RINGS

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Abstract. We show that there is a solution $f$ of the equation

$$f(x) = f(a) + (RL) \int_a^x (fH + fG)$$

such that $f(p) = 0$ and $f(q) \neq 0$ for some pair $p, q \in [a, b]$ iff there is a number $t \in [a, b]$ such that one of $1 - H(t, t), 1 - H(t, t^+), 1 + G(t, t)$ or $1 + G(t, t^+)$ is zero or a right divisor of zero, where $f, G$ and $H$ are functions of bounded variation with ranges in a normed ring $N$. Furthermore, if $N$ is a field, then for each discontinuity of $H$ on $[a, b]$ there exists $\lambda \in N$ and a finite set of linearly independent nonzero solutions on $[a, b]$ of the equation $f(x) = f(a) + (RL) \int_a^x (fH + fG)$. If $f$ is a solution and has bounded variation on $[a, b]$, then $f$ is a linear combination of this set of solutions. Product integrals are used extensively in the proofs.

1. Definitions and preliminary theorems. For detailed definitions see [1, p. 299]. $R$ is the set of real numbers, $N$ is a ring which has a multiplicative identity element 1 and a norm $|\cdot|$ with respect to which $N$ is complete and $|1| = 1$; $G$ and $H$ are functions from $R \times R$ to $N$ and functions from $R$ to $N$ are denoted by lower case letters. The symbol $<$ is defined by one of the following statements: (1) if $x$ and $y \in R$, then $x < y$ iff $y$ is less than $x$, and (2) if $x$ and $y \in R$, then $x < y$ iff $x$ is less than $y$. The symbols $[x, y], G(x, y), \int_a^b, \int_0^\infty$, etc. imply that $x < y$. $\{x_i\}_{i=0}^n$ is a subdivision of $[q, p]$ means $q = x_0 < x_1 < \cdots < x_n = p$. All sum and product integrals (represented by $\int_a^b G$) are subdivision-refinement-type limits; $\int_0^\infty G = 1$,

$$(RL) \int_a^b (fH + fG) \sim f(y)H(x, y) + f(x)G(x, y)$$

and

$$(m) \int_a^b fH \sim \frac{1}{2} [f(x) + f(y)]H(x, y) \quad \text{for } a \leq x < y \leq b.$$
has bounded variation on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. $G \in OI^0$ on $[x, y]$ means there is a subdivision $\{x_i\}_0^n$ of $[x, y]$ such that if $0 < i \leq n$ then the multiplicative inverse of $G$ exists and is bounded on $[x_{i-1}, x_i]$; appropriate modifications are used for open and half open intervals. $G \in OL^0$ on $[a, b]$ iff $\lim_{x \to p^-} G(x, p)$ and $\lim_{y \to p^+} G(x, y)$ exist for $p \in (a, b)$ and $\lim_{x \to p^-} G(p, x)$ and $\lim_{y \to p^+} G(x, y)$ exist for $p \in [a, b)$. When confusion is unlikely, phrases such as “on $[a, b]$” will be omitted, $(RL)^*_a(fH+fG)$ will be denoted by $J_a\{ fH+fG \}$, and “the given equation on $[x, y]$” refers to the equation $f(y) = f(x) + (RL)^*_a(fH+fG)$.

**Theorem 1.** If $H$ and $G$ are functions from $R \times R$ to $N$ such that $G \in O^A$ and $O^B$ and $H \in OL^0$ on $[a, b]$, then $GH$ and $HG \in O^A$ and $OM^0$ on $[a, b]$.

This is Theorem 2 in [2, p. 494].

If $H$ and $G \in O^A$ and $O^B$ and $f$ has bounded variation on $[a, b]$, it follows from Theorem 3.5 [1, p. 303] that $f(y)H(x, y) + f(x)G(x, y) \in O^A$ on $[a, b]$.

**Theorem 2.** If $H$ and $G$ are functions such that $H$ and $G \in O^A$ and $O^B$ and $(1 - H) \in OI^0$ on $[a, b]$, then $z \prod^{\nu}(1 + G)(1 - H)^{-1}$ exists for $a \leq x < y \leq b$ and, if $f$ is a function, the following statements are equivalent.

1. $f(y)H(x, y) + f(x)G(x, y) \in O^A$ and $f(x) = f(a) + (RL)^*_a(fH+fG)$ for $a \leq x \leq b$.

2. If $a \leq x < y \leq b$, then $f(y) = f(x)$, $z \prod^{\nu}(1 + G)(1 - H)^{-1}$.

**Proof.** Let $A = (1 + G)(1 - H)^{-1}$, then $A - 1 = (H + G)(1 - H)^{-1} \in O^B$. Since $H \in O^B$ and $1 - H \in OI^0$, then $(1 - H)^{-1} \in OL^0$ and, by Theorem 1, $A - 1 \in OM^0$ and $(1 - H)^{-1} \in O^B$. Furthermore, if $f$ is bounded, then $(L)^*_a \{ f( ) \prod A - A ( , ) \} = 0$. It follows from Theorem 5.1 [1, p. 310] that the two statements are equivalent. Note that $A \prod^{\nu} A$ is a bounded function.

2. **Principal theorems.** A corollary to the following theorem is obtained by using the conditions enclosed by brackets in place of those in quotation marks.

**Theorem 3.** If $H$ and $G \in O^A$ and $O^B$ on $[a, b]$, the following statements are equivalent:

1. There is a function $f$ and numbers $p$ and $q$ such that $a \leq p < q \leq b$, “$f(p) = 0$” [f(p) $\neq 0$], “$f(q) \neq 0$” [f(q) = 0], $f$ has bounded variation on $[p, q]$ and $f(x) = f(p) + (RL)^*_p(fH+fG)$ for $x \in (p, q)$.

2. There is a number $t \in [a, b]$ such that $t \neq a$ and “$1 - H(t-, t)$” [1 + $G(t-, t)$] is zero or a right divisor of zero or such that $t \neq b$ and “$1 - H(t, t^+)$” [1 + $G(t, t^+)$] is zero or a right divisor of zero.
PROOF. (1)→(2). Let S be the number set such that \( x \in S \) iff \( x \in [p, q] \) and \( f(x) = 0 \); then S has a least upper bound \( t \) and \( p \leq t \leq q \). If \( f(t) \neq 0 \), then \( p \leq p < t \),

\[
f(t) = f(p) + \int_p^t fH + fG = f(t^-) + f(t)H(t, t)
\]

and

\[
0 = f(t^-) = f(t)[1 - H(t, t)]
\]

and therefore \( 1 - H(t, t) \) is zero or a right divisor of zero.

If \( f(t) = 0 \), then \( t < q \leq b \) and

\[
f(t^+) = f(p) + \int_p^t fH + fG = f(t) + \int_t^{t^+} fH + fG = f(t^+)H(t, t^+)
\]

and therefore \( f(t^+) \left[ 1 - H(t, t^+) \right] = 0 \) and \( 1 - H(t, t^+) \) is zero or a right divisor of zero, provided \( f(t^+) \neq 0 \). Suppose \( f(t^+) = 0 \); then there is a number \( c \) such that \( t < c < q \) and \( 1 - H \in OI^0 \) on \( (t, c) \); hence, by Theorem 2, if \( x \in (t, c) \), then

\[
f(x) = f(t^+) + \int_t^x fH + fG = f(t^+) \int_t^x (1 + G)(1 - H)^{-1} = 0.
\]

Therefore, \( t \) is not the least upper bound of \( S \).

(2)→(1). Suppose \( t \neq a \) and \( 1 - H(t, t) \) is zero or a right divisor of zero; let \( p = a, q = t, k \) be a nonzero element of \( N \) such that \( k[1 - H(t, t)] = 0 \), and let \( f \) be the function such that \( f(x) = 0 \) for \( x \in [a, t) \) and \( f(t) = k \). If \( x \in [a, q) \), then \( f(a) + \int_a^x fH + fG = 0 = f(x) \). If \( x = q = t \), then

\[
f(a) + \int_a^t fH + fG = f(t^-) + \int_t^t fH + fG = f(t)H(t, t)
\]

\[
= f(t) - k[1 - H(t, t)] = f(t) = f(x).
\]

Suppose \( t \neq b \) and \( 1 - H(t, t^+) \) is zero or a right divisor of zero. There is a number \( q \) such that \( t < q < b \) and such that \( 1 - H \in OI^0 \) on \( (t, q) \). Also, there is a nonzero element \( k \in N \) such that \( k[1 - H(t, t^+)] = 0 \). Let \( p = a \) and define \( f \) to be the function such that \( f(x) = 0 \) for \( x \in [a, t] \) and

\[
f(x) = k_{t^+} \prod_{x}^x (1 + G)(1 - H)^{-1} \quad \text{for} \quad x \in (t, q];
\]

then \( f(t^+) = k \). If \( x \in (t, q] \), then
\[
f(p) + \int_{p}^{x} fH + fG = \left( \int_{t}^{t^+} + \int_{t^+}^{x} \right) (fH + fG) = f(t^+)H(t, t^+) + \int_{t^+}^{x} fH + fG
\]
\[
= k - k[1 - H(t, t^+)] + \int_{t^+}^{x} fH + fG = k + \prod_{t}^{x} (1 + G)(1 - H)^{-1}
\]
\[
= f(x).
\]

Since \(H\) and \(G \in OB^o\) and \(f\) is bounded on \([p, q]\), then \(\int fH + fG\) and \(f\) have bounded variation on \([p, q]\).

**Proof of corollary.** Since \(f(y) = f(x) + (RL) \int_{x}^{y} fH + fG\) for \(a \leq x < y \leq b\), then \(f(\alpha) = f(\beta) + (RL) \int_{\alpha}^{\beta} fH + fG\) for \(a \leq \alpha < \beta \leq b\).

Proof. Since \(f(x) = f(x) - (RL) \int_{x}^{y} fH + fG = f(x) + (RL) \int_{x}^{y} fH + fG = f(x) + (RL) \int_{x}^{y} f(-g) + f(-h),\)

where \(g(y, x) = G(x, y)\) and \(h(y, x) = H(x, y)\), it follows that this corollary is a special case of the preceding theorem with \(-g\) and \(-h\) playing the roles of \(H\) and \(G\), respectively.

**Lemma.** If \(f(x) = f(a) + (RL) \int_{a}^{x} fH + fG\) for \(x \in [a, b]\), then
\[
(1) \text{ if } x \in (a, b), \ f(x)[1 - H(x, x)] = f(x)[1 + G(x, x)], \text{ and}
\]
\[
(2) \text{ if } x \in [a, b), \ f(x^+)[1 - H(x, x^+)] = f(x)[1 + G(x, x^+)].
\]

**Theorem 4.** Given: \(a \leq p \leq b; H\) and \(G \in OA^o\) and \(OB^o\) on \([a, b]\); if \(a < p\), then \(H(p, p) = 1\); if \(p < b\), then \(H(p, p^+) = 1\) and \(1 + G(p, p^+)\) is not a right divisor of zero; there is a function \(f\) of bounded variation on \([a, b]\) such that \(f(p) \neq 0\) and \(f(x) = f(a) + (RL) \int_{a}^{x} fH + fG\) for \(x \in [a, b]\); and \(u\) is a function such that \(u(x) = 0\) if \(x \neq p\).

**Conclusion.** If \(x \in [a, b]\), then \(u(x) = u(a) + (RL) \int_{a}^{x} (uH + uG)\).

Proof. If \(a < p \leq b\), then \(u(x) = 0\) for \(a \leq x < p\) and
\[
u(a) + \int_{a}^{p} uH + uG = u(p)H(p, p) = u(p).
\]

If \(a \leq p < b\), then it follows from the lemma that
\[
f(p)[1 + G(p, p^+)] = f(p^+)[1 - H(p, p^+)] = 0;
\]
hence, \(1 + G(p, p^+) = 0\) and, if \(x \in (p, b]\), then
\[
u(p) + \int_{p}^{x} uH + uG = u(p) + u(p^+)H(p, p^+) + u(p)G(p, p^+)
\]
\[
= u(p)[1 + G(p, p^+)] = 0 = u(x).
\]
If \( a < p < b \), it follows from the two preceding results that \( u(x) = u(a) + (RL) \int_a^x (uH + uG) \) for \( x \in [a, b] \).

In the following theorems the symbol \( \langle p, q \rangle \) denotes a subset of \( [a, b] \) such that

1. \( 1 - H \in OI^0 \) on \( \langle p, q \rangle \) and \( \langle p, q \rangle \subseteq \langle p, q \rangle \subseteq \langle p, q \rangle \);
2. \( H \) has a discontinuity of 1 at \( p \) provided \( p \neq a \), and at \( q \) provided \( q \leq b \); and
3. \( p \in \langle p, q \rangle \) iff \( H(p, p^+) \neq 1 \), and \( q \in \langle p, q \rangle \) iff \( H(q^-, q) \neq 1 \). Also, if \( p \in \langle p, q \rangle \) and \( a \neq p \), then \( p' \), \( p^* \) denotes \( p^- \), \( p \); if \( p \in \langle p, q \rangle \), then \( p', p^* \) denotes \( p, p^+ \); if \( q \in \langle p, q \rangle \), then \( q', q^* \) denotes \( q^- \), \( q \); and if \( q \in \langle p, q \rangle \), then \( q', q^* \) denotes \( q^-, q^+ \); if \( a \in \langle p, q \rangle \), the \( p^* = a \).

**Theorem 5.** Given, \( a \leq p < q \leq b \); \( H \) and \( G \in OA^0 \) and \( OB^0 \); \( (1 - H) \in OI^0 \) on \( \langle p, q \rangle \); either \( a \in \langle p, q \rangle \), or \( a < p \in \langle p, q \rangle \) and \( H(p, p) = 1 \), or \( a \leq p \in \langle p, q \rangle \) and \( H(p, p^+) = 1 \); either \( b \in \langle p, q \rangle \), or \( b \leq q \in \langle p, q \rangle \) and \( H(q^-, q) = 1 \), or \( b > q \in \langle p, q \rangle \) and \( H(q^-, q^*) = 1 \); if \( x \in [a, b] \), then neither \( 1 + G(x^-, x) \) or \( 1 + G(x, x^*) \) is a right divisor of zero; there is a function \( f \) with bounded variation on \( [a, b] \) and a number \( t \in \langle p, q \rangle \) such that \( f(t) \neq 0 \) and \( f(x) = f(a) + (RL) \int_a^x (fH + fG) \) for \( x \in [a, b] \); \( u \) is a function such that \( u(x) = 0 \) for \( x \in \langle p, q \rangle \), \( u(p^*) = 1 \), and if \( x \in \langle p, q \rangle \) then \( u(x) = u(a) + (RL) \int_a^x (uH + uG) \).

**Conclusion.** (1) If \( x \in [a, b] \), then \( u(x) = u(a) + (RL) \int_a^x (uH + uG) \).

(2) If the function \( w \) has bounded variation and is a solution of the given equation on \( [a, b] \), then \( w(x) = w(p^*) u(x) \) for \( x \in \langle p, q \rangle \) and, if there exists a number \( c \in \langle p, q \rangle \) such that \( w(c) \neq 0 \), then \( w(p^*) \neq 0 \).

**Proof of (1).** If \( a \in \langle p, q \rangle \), it follows from the preceding theorem that \( u \) is a solution on \( [a, p^*] \); hence, if \( x \in \langle p, q \rangle \), then

\[
\begin{align*}
u(a) + \int_a^x uH + uG &= u(p^*) + \int_{p^*}^x uH + uG \\
&= u(p^*) \prod_{p^*}^x (1 + G) (1 - H)^{-1} = u(x).
\end{align*}
\]

Suppose \( b \in \langle p, q \rangle \). It follows from Theorem 2 that \( u \) is a solution on \( \langle p, q \rangle \); hence, if \( x \in [q^*, b] \), then \( u(q^*) = 0 \) and

\[
\begin{align*}
u(p^*) + \int_{p^*}^x uH + uG &= u(q^*) + \left( \int_{q^*}^x + \int_{q^*}^{q^*} \right) (uH + uG) \\
&= u(q^*) + u(q^*) H(q^*, q^*) + u(q^*) G(q^*, q^*) \\
&= u(q^*) \left[ 1 + G(q^*, q^*) \right] = 0 = u(x),
\end{align*}
\]

provided one of \( u(q^*) \) or \( 1 + G(q^*, q^*) \) is zero.
In order to show that the preceding requirement is satisfied, we will consider two cases: \( f(q') \neq 0 \) and \( f(q') = 0 \). If \( f(q') \neq 0 \), then

\[
0 = f(q') [1 - H(q', q^*)] = f(q') [1 + G(q', q^*)]
\]

and \( 1 + G(q', q^*) = 0 \) because \( 1 + G(q', q^*) \) is not a right divisor of zero. If \( f(q') = 0 \), then it follows from the corollary to Theorem 3 that there is a number \( z \) such that \( t < z \leq q' \) and such that \( 1 + G(z', z^*) = 0 \); hence, \( u(q') = p \prod q' (1 + G)(1 - H)^{-1} = 0 \).

If \( a \in \langle p, q \rangle \) and \( b \in \langle p, q \rangle \), it follows from the two preceding results that

\[
u(x) = u(a) + (RL) \int_a^x (uH + uG) \quad \text{for} \quad x \in [a, b].
\]

Let \( c \) be a number such that \( p < c < q \); then, if \( q^* \leq x \leq b \), it follows that

\[
u(a) + \int_a^x uH + uG = u(c) + \int_c^x uH + uG = u(x).
\]

Proof of (2). If \( x \in \langle p, q \rangle \), then, by Theorem 2,

\[
u(x) = \frac{w(x)}{w(\langle p \rangle)} + \int_a^x wH + wG = w(p^*) + \int_{p^*}^x wH + wG
\]

\[
= w(p^*)p^* \prod q'(1 + G)(1 - H)^{-1} = w(p^*)u(x)
\]

and \( w(x) = 0 \) if \( w(p^*) = 0 \). Hence, if \( c \in \langle p, q \rangle \) and \( w(c) \neq 0 \), then \( w(p^*) \neq 0 \).

Theorem 6. Given. \( H \) and \( G \) are functions from \( R \times R \) to \( N \) such that \( H \) and \( G \in OA^0 \) and \( OB^0 \) on \( [a, b] \); if \( x \in \langle a, b \rangle \) and \( 1 - H(x^-, x)^{-1} \) does not exist, then \( H(x^-, x) = 1 \); if \( x \in \langle a, b \rangle \) and \( \left[ 1 - H(x, x^+) \right]^{-1} \) does not exist, then \( H(x, x^+) = 1 \); if \( x \in \langle a, b \rangle \), then neither of \( 1 + G(x^-, x) \) or \( 1 + G(x, x^+) \) is a right divisor of zero.

Conclusion. There is a finite set of linearly independent solutions of the equation \( f(x) = f(a) + (RL) \int_a^x (fH + fG) \) on \( [a, b] \) such that a function \( f \) is a linear combination of this set iff \( f \) has bounded variation on \( [a, b] \) and \( f \) is a solution to the given equation on \( [a, b] \).

Proof. It is assumed that the equation has at least one nonzero solution. Let \( \{x_i\}^n_0 \) be the subdivision of \( [a, b] \) such that \( x \in \{x_i\}^{n-1} \) if \( H(x^-, x) = 1 \) or \( H(x, x^+) = 1 \); then \( (1 - H) \in O^0 \) on \( \langle x_{i-1}, x_i \rangle \) for \( i = 1, 2, \ldots, n \).

Let \( P \) be the set of integers such that \( i \in P \) iff \( 0 < i \leq n \) and there is a solution \( f \) on \( [a, b] \) and a number \( t \in \langle x_{i-1}, x_i \rangle \) such that \( f(t) \neq 0 \). For
each \( i \in P \), define \( c_i = x_{i-1}^{*} \) and \( u_i \) to be the function defined in Theorem 5, where \( c_i \) corresponds to \( p^{*} \) and \( u_i (c_i) = 1 \).

Let \( O \) be the set of integers such that \( i \in Q \) iff \( 0 \leq i \leq n \) and \( x_i \in \cup_{i \in P} (x_{i-1}, x_i) \) and there is a solution \( f \) on \([a, b]\) such that \( f(x_i) \neq 0 \). For \( i \in Q \), define \( w_i \) to be the function such that \( w_i (x_i) = 1 \) and \( w_i (x) = 0 \) if \( x \neq x_i \); it follows from Theorem 4 that \( w_i \) is a solution of the equation on \([a, b]\).

The set \( \{ u_i \}_{i \in P} \cup \{ w_i \}_{i \in Q} \) of functions is the desired set. Since each function belonging to the set has bounded variation and is a solution of the equation on \([a, b]\), then each linear combination of these functions is a solution and has bounded variation on \([a, b]\). If \( \{ k_i \}_{i \in P} \) and \( \{ h_i \}_{i \in Q} \) are subsets of \( N \) and if \( m \in P \), then

\[
\sum_{i \in P} k_i u_i (c_m) + \sum_{i \in Q} h_i w_i (c_m) = k_m u_m (c_m) = k_m;
\]
a similar result holds for \( m \in Q \). Therefore, if

\[
\sum_{i \in P} k_i u_i (x) + \sum_{i \in Q} h_i w_i (x) = 0
\]

for all \( x \in [a, b] \), then each of the coefficients is zero; hence, the functions are linearly independent.

If \( f \) is a solution of the equation and has bounded variation on \([a, b]\) and \( x \in [a, b] \), then the summation

\[
\sum_{i \in P} f(c_i) u_i (x) + \sum_{i \in Q} f(x_i) w_i (x) = g(x)
\]
simplifies as follows:

1. If \( i \in P \) and \( x \in (x_{i-1}, x_i) \), then \( g(x) = f(c_i) u_i (x) = f(x) \), by Theorem 5;
2. If \( i \in Q \) and \( x = x_i \), then \( g(x) = f(x_i) w_i (x) = f(x_i) = f(x) \); and
3. If \( x \in (x_{i-1}, x_i) \) and \( i \in P \) or if \( x = x_i \) and \( i \in Q \), then \( g(x) = 0 \).

From the definition of \( P \) and \( Q \), if the conditions in (3) are satisfied, then \( f(x) = 0 \). Hence, the above summation is \( f(x) \) for \( x \in [a, b] \).

3. **Comments.** If \( N \) is a field and \( H \) and \( G \in OA^{0} \) and \( OB^{0} \) on \([a, b]\), then each of the equations

\[
f(x) = (RL) \int_{a}^{x} (fH + fG) \lambda, \quad f(x) = (R) \int_{a}^{x} fH \lambda
\]

and

\[
f(x) = (m) \int_{a}^{x} fH \lambda
\]
has a solution on \([a, b]\) iff \(H\) has a discontinuity \(k\) on \([a, b]\), in which case \(\lambda = k^{-1}\). If \(k\) is such a discontinuity of \(H\), then there is a largest number \(p \in [a, b]\) at which the discontinuity occurs and the function \(f\) can be defined on \([a, b]\) as in Theorem 3(2)\(\rightarrow\)(1). The set of \(\lambda\)'s may be infinite but cannot be uncountable. The possibility that the equation 
\[f(x) = f(a) + (RL) \int_a^x (fH + fG),\]
has a solution \(f\) on \([a, b]\) for which \(f(a) \neq 0\) depends on the order of occurrence and relative values of the discontinuities of \(H\) and \(G\).

The following conjectures are probably true.

1. Similar theorems will hold for the equations

\[f(x) = f(a) + (RL) \int_a^x (Hf + Gf),\]

\[f(x) = f(a) + (RL) \int_a^x (fH + Gf)\]

and

\[f(x) = f(a) + (RL) \int_a^x (Hf + fG).\]

2. The set \(R\) can be any linearly ordered set \([4, \text{p. 149}]\).
3. In Theorems 4 and 5 the restrictions on \(1-H\) can be relaxed to permit \(1-H(x^-, x)\) and \(1-H(x, x^+)\) to be right divisors of zero.

Bibliography


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