ON THE EXISTENCE OF $L_{\infty\kappa}$-INDISCERNIBLES

P. C. EKLOF

Abstract. It is proved that if $T$ is a countable theory of $L_{\omega_1\omega}$ with enough axioms for Skolem functions and with arbitrarily large models, then for any order type, there is a model of $T$ with a set of $L_{\infty\kappa}$-indiscernibles of that order type.

In this short note we answer in the affirmative a question of Chang [4] as to whether there exist $L_{\infty\kappa}$-indiscernibles of any given order type. In fact we prove a somewhat stronger result since we show the existence of $L_{\infty\kappa}$-indiscernibles and we use a stronger definition of indiscernibles. Our result also gives a simpler proof of Chang’s theorem on the existence of $L_{\kappa\omega}$-indiscernibles of any well-ordered order type [4, Theorem 4]. (We thank Jon Barwise for some helpful discussions.)

In general we follow the notion of Chang [4] (so $\kappa$ is always an infinite regular cardinal). Let $L$ be a first order language with countably many relation, function, and constant symbols, and let $\mathfrak{A} = (A, \cdots)$ be a structure for $L$. An ordered subset $X$ of $A$ is said to be $L_{\lambda\kappa}$-indiscernible if for any subset $Y$ of $X$ of cardinality $\lambda$ and any order-preserving injection $h: Y \to X$,

$$(\mathfrak{A}, y)_{y \in Y} \equiv_{\lambda} (\mathfrak{A}, hy)_{y \in Y}.$$

If $L_{\infty\kappa}$ is the union of the infinitary languages $L_{\lambda\kappa}$, where $\lambda$ ranges over all cardinals, we define $L_{\infty\kappa}$-indiscernibles analogously.

Let $T$ be a countable theory of $L_{\omega_1\omega}$. There is a countable fragment $\mathcal{L}_A$ such that $T \subseteq \mathcal{L}_A$ (for the definition of $\mathcal{L}_A$ see Barwise [1]). We consider only $T$ and $\mathcal{L}_A$ such that $\mathcal{L}_A$ has enough function symbols and $T$ includes axioms for all Skolem functions of formulas of $\mathcal{L}_A$. A necessary condition for $T$ to have models with sets of $L_{\infty\kappa}$-indiscernibles of any order type is that $T$ have models of arbitrarily large cardinality; this is sufficient as well.

Theorem. Let $T \subseteq \mathcal{L}_A$ such that $T$ has arbitrarily large models. If $\mu$ is any order type, there is a model $\mathfrak{A}$ of $T$ such that $\mathfrak{A}$ has a set of $L_{\infty\kappa}$-indiscernibles of order type $\mu$.

Proof. We may suppose that the cardinality $|\mu|$ of $\mu$ is $\leq \kappa$, since a set of $L_{\infty\lambda}$-indiscernibles is a set of $L_{\infty\kappa}$-indiscernibles if $\lambda \geq \kappa$. Sup-

Received by the editors December 7, 1969.

AMS Subject Classifications. Primary 0235.

Key Words and Phrases. Indiscernibles, infinitary languages, $\eta_{\omega}$-set.

798
pose \( \kappa = \aleph_\alpha \); it suffices to prove that \( T \) has a model \( \mathfrak{A} \) with a set \( X \) of \( L_\omega \)-indiscernibles of order type \( \eta_\alpha \), since \( \mu \) can be embedded in \( X \) [7, pp. 334–338].

We are assuming that models of \( T \) have Skolem functions for all formulas of \( L_\omega \). Since \( T \) has arbitrarily large models, there is a model \( \mathfrak{A} \) of \( T \) with a set \( X \) of \( \omega \)-indiscernibles of order type \( \eta_\alpha \) (see [6]; if \( L_\omega = L_{\omega_1} \) this is just the classical result of Ehrenfeucht-Mostowski [5]). We may suppose that \( \mathfrak{A} = \mathfrak{S}(X) \), where \( \mathfrak{S}(X) \) is the Skolem hull of \( X \) (i.e. the submodel of \( \mathfrak{A} \) whose universe \( A \) is the closure of \( X \) under the Skolem functions of \( L_\omega \)).

We claim that \( X \) is a set of \( L_\omega \)-indiscernibles in \( \mathfrak{A} \). Let \( Y \subseteq X \) be of cardinality \( \kappa = \aleph_\alpha \) and let \( h : Y \to X \) be an order-preserving injection. Let \( I \) be the set of all isomorphisms

\[ f : S \to S' \]

of submodels \( S, S' \) of \( \mathfrak{A} \) such that \( Y \subseteq S, f|_Y = h \), and there exist \( U, U' \subseteq X \) such that \( |U| < \kappa, S = \mathfrak{S}(U), S' = \mathfrak{S}(U') \) and \( f|_U \) is an order-isomorphism of \( U \) onto \( U' \). Notice that \( I \neq \emptyset \) since, letting \( S = \mathfrak{S}(Y), S' = \mathfrak{S}(h(Y)) \), there is an extension of \( h \) to an isomorphism \( f : S \to S' \).

We claim that \( I \) has the following property:

\[ (*) \]

For any \( C \subseteq A \) such that \( |C| < \kappa \) and any \( f \in I \), there are \( f', f'' \in I \) such that \( f \subseteq f', f \subseteq f'' \), \( C \subseteq \text{domain of } f' \), and \( C \subseteq \text{range of } f'' \).

It suffices to prove \( (*) \), for it follows easily by an induction on formulas of \( L_\omega \) that

\[ (\mathfrak{A}, y)_{y \in Y} \equiv_{\omega_\alpha} (\mathfrak{A}, hy)_{y \in Y} \]

(see Calais [2]).

So suppose \( f : S \to S' \) is in \( I \) and \( U, U' \) are as in the definition of \( I \). Given \( C \subseteq A \), \( |C| < \kappa \), there is a \( D \subseteq X \), \( |D| < \kappa \), such that \( C \subseteq \mathfrak{S}(U \cup D) \). It is clear that in order to prove the existence of \( f' \) as required by \( (*) \), it suffices to show that we can extend \( f|_U : U \to U' \) to an order-monomorphism: \( U \cup D \to X \). We may assume \( D \cap U = \emptyset \). Define an equivalence relation on \( D \) by: \( x \sim y \) iff \( x \) and \( y \) determine the same cut of \( U \). Write \( D = \bigcup_{y \in \tau} D_x \) as the union of pairwise disjoint equivalence classes \( D_x, \sigma < \tau < \kappa \). For any \( \sigma < \tau \), let \( U = A_x \cup B_x \) where \( A_x < D_x < B_x \). Then \( f(A_x) < f(B_x) \) and \( |f(A_x)| < \kappa, |f(B_x)| < \kappa \). So if

\[ E_\sigma = \{ x \in X : f(A_x) < x < f(B_x) \} , \]
$E_\alpha$ is an $\eta_\alpha$-set. Therefore there exists an embedding

$$g_\alpha : D_\alpha \to E_\alpha.$$  

Define $f' : U \cup D \to X$ by: $f'(x) = f(x)$ if $x \in U$; $f'(x) = g_\alpha(x)$ if $x \in D_\alpha$.

This gives the desired extension of $f$. In a similar manner we can prove the existence of $f''$ extending $f$ with $C \subseteq$ range of $f''$. This completes the proof.

**Remarks.** (1) If we assume the generalized continuum hypothesis then the proof is much simpler; for then there exists an $\eta_\alpha$-set $X$ of cardinality $\mathfrak{N}_\alpha$. Hence if $h : Y \to X$ is an order-preserving injection and $|Y| < \kappa$, $h$ extends to an isomorphism $h' : X \to X$. It is immediate that

$$(\mathcal{A}, y)_{y \in Y} = \omega_\kappa (\mathcal{A}, hy)_{y \in Y}.$$  

(Compare the remark of Chang [3, p. 55].)

(2) Our method suffers from the same defect as that of Chang, namely the indiscernibles do not necessarily generate the model.

(3) If $\kappa = \mathfrak{N}_\alpha$ and $\kappa \geq |\mu|$ the model $\mathcal{A}$ asserted to exist in the statement of the theorem can be chosen to have cardinality $= 2^{\mathfrak{N}_\beta}$ if $\alpha = \beta + 1$; $\sum_{\beta < \alpha} 2^{\mathfrak{N}_\beta}$ if $\alpha$ is a limit ordinal [7].

**References**


Yale University, New Haven, Connecticut 06520