LET F be a compact set of real numbers and [a, b] the smallest interval containing F. The complement [a, b] \sim F is composed of a countable sequence of disjoint open intervals, of lengths \( l_n \). We investigate sets F of Lebesgue measure 0 with the property that \( \sum l_n^\beta < \infty \) for some \( c \) in (0, 1). These sets were considered by Besicovitch and Taylor in [1] but our theorems are in a different direction. We require a class of functions \( C^\beta \) defined for each number \( \beta > 1 \): a real function \( f \) on an interval is of class \( C^\beta \) provided it is \( n \) times continuously differentiable, where \( n < \beta \leq n + 1 \), and \( D^n f \) is of class \( \text{Lip}^{\beta-n} \). When \( \beta = n + 1 \) this conflicts with the usual definition of \( C^{n+1} \), but no confusion is to be expected; in fact by allowing a larger class \( C^{n+1} \) we obtain a slightly sharper result.

**Theorem 1.** Let \( f \) belong to \( C^\beta \), let \( Z \) be the zero-set of \( Df \), and let \( F = f(Z) \). Then \( F \) has Hausdorff \( 1/\beta \)-measure 0, and the lengths \( l_n \) fulfill the condition \( \sum l_n^{1/\beta} < \infty \).

**Theorem 2.** Conversely, let \( F \) be a compact set of Lebesgue measure 0, whose contiguous intervals fulfill the convergence condition above. Then \( F = f(Z) \) for some function \( f \) in \( C^\beta \) for which \( Df \geq 0 \) and whose zero-set \( Z \) has Lebesgue measure 0. When \( \beta = n + 1 \), \( f \) can be made \( n + 1 \) times continuously differentiable.

**Notation.** The diameter of a set \( E \) is written \( |E| \), and its Lebesgue measure \( m(E) \). The modulus of continuity of a function \( f \) on a set \( T \) is defined for \( u > 0 \) as

\[
w(u) = \sup |f(t_1) - f(t_2)| : |t_1 - t_2| \leq u.
\]

Then \( w(u) \ll u^c \) defines the class \( \text{Lip}^c \), \( 0 < c \leq 1 \).

1. The proof of Theorem 1 is largely a variant of Taylor's theorem,
the object being to exploit the extra information on the highest-order derivative.

**Lemma 1.** Let $f$ be $k$ times continuously differentiable on an interval $[c, d]$ and let $Df, \ldots, D^k f$ vanish at least once in the interval. Then

$$
\int_c^d |Df(t)| \, dt \leq (d - c)^{k-1} \int_0^{d-c} w(u) \, du,
$$

where $w$ is the modulus of continuity of $D^k f$.

**Proof.** For $k = 1$ and $Df(\xi) = 0$, $c \leq \xi \leq d$, we have

$$
\int_c^d |Df(t)| \, dt \leq \int_0^{\xi-d} + \int_0^{d-\xi} w(u) \, du \leq \int_0^{d-c} w(u) \, du.
$$

Assuming the truth of the lemma for $k - 1 \geq 1$,

$$
\int_c^d |D^2f(t)| \, dt \leq (d - c)^{k-2} \int_0^{d-c} w(u) \, du.
$$

Because $Df$ has a zero in $[c, d]$, $|Df| \leq \int_c^d |D^2f(t)| \, dt$ and the lemma follows from this.

To prove Theorem 1, we observe first that each interval $I = (t_1, t_2)$ contiguous to $f(Z)$ has the form $f(J)$ for some interval $J$ contiguous to $Z$. Indeed, let $s_1 \in f^{-1}(t_1)$ and $s_2 \in f^{-1}(t_2)$ be so chosen that $|s_1 - s_2|$ attains its minimum value. Then the interval $J$ between $s_1$ and $s_2$ is mapped into $I$, and therefore onto $I$. Thus it is sufficient to prove that

$$
\sum |f(J)|^{1/\beta} < \infty,
$$

where the summation is extended to intervals $J$ contiguous to $Z$.

First, let $Z'$ be the derived set of $Z$ and let $J$ have at least one end point in $Z'$. Then $Df, \ldots, D^n f$ vanish there and by Lemma 1

$$
|f(J)| = \left| \int_J Df \right| \leq |J|^{n-1} \int_0^{|J|} w(u) \, du \ll |J|^\beta.
$$

To treat the isolated points in $Z$, let $J$ be an interval contiguous to $Z'$, so that if $J$ meets $Z$ then $J \cap Z$ is discrete. Thus $J \cap Z$ can be enumerated $\cdots < z_{-1} < z_0 < z_1 < \cdots$, and we must estimate the sum $\cdots + |f(z_1) - f(z_0)|^{1/\beta} + \cdots$. The sequence $z_{-1} < z_0 < z_1 < \cdots$ can be arranged into disjoint blocks of exactly $n+1$ terms, with a possible remainder of at most $n$ terms. By Rolle's Theorem we know that $D^2f, \ldots, D^n f$ each have zeros on any interval $[z_i, z_{i+n}]$, whence

$$
\int_{z_{i+n}}^{z_{i+n}} |Df| \ll |z_{i+n} - z_i|^\beta.
$$

The same estimate can be made for the remainder allowed before, because one of the extreme terms is succeeded immediately by an element of $Z'$. Applying the inequality
\[
\sum_{i=0}^{n} x_i^{1/\beta} \leq (n + 1)^{1-1/\beta} \left( \sum_{i=0}^{n} x_i \right)^{1/\beta},
\]
we find that
\[
\cdots + |f(z_1) - f(z_0)|^{1/\beta} + \cdots \ll |J|.
\]

The estimation given applies to all but 2\(n\) intervals situated entirely to one side of \(Z'\), and the proof is complete. (The possibility that \(Z' = \emptyset\) makes the last remark necessary.)

That \(F\) has Hausdorff \(1/\beta\)-measure 0 is proved very simply in [1], but we present a less elementary proof for a stronger conclusion.

**Lemma 2.** Let \(f\) be absolutely continuous on an interval \([c, d]\) and let \(\int_A |Df| = 0\) for a closed subset \(A \subseteq [c, d]\). Suppose that for every interval \(J\) contiguous to \(A\),
\[
|Df| \ll |J|^\beta
\]
for a certain real number \(\beta > 1\).

Then \(f(A)\) is contained in \(o(N)\) intervals of length \(N^{-\beta}\), \(N \to +\infty\).

**Proof.** We shall replace \(f\) by a function \(g\) that coincides with \(f\) on \(A\), and is again absolutely continuous. To do so we define \(Dg\) on the intervals \(J\) so that \(\int_J Dg = \int_J Df\). Thus, when \(J = (t_1, t_2)\), set
\[
Dg(s) = c(s - t_1)^{\beta-1}, \quad t_1 < s \leq \frac{1}{2}(t_1 + t_2),
\]
\[
Dg(s) = c(t_2 - s)^{\beta-1}, \quad \frac{1}{2}(t_1 + t_2) < s < t_2
\]
for a constant \(c\). Then in fact
\[
c = 2^{\beta-1}(t_2 - t_1)^{-\beta}(f(t_2) - f(t_1)) \ll 1,
\]
whence
\[
Dg(s) \ll \Theta(\text{dist}(s, A))^{\beta-1}.
\]

Let \(I_N\) denote any of the intervals \([kN^{-1}, (k+1)N^{-1}]\) that meet \(A\), so that \(f(A) = g(A) \subseteq \bigcup g(I_N)\). On each \(I_N\) we have \(|Dg| \ll N^{1-\beta}\), so that \(|g(I_N)| \ll N^{-\beta}\). Of course, the number of intervals \(I_N\) is \(O(N)\).

Fixing a number \(\varepsilon < 1\) we divide the intervals \(I_N\) into two classes.

(i) \(m(I_N \cap A) > (1-\varepsilon)N^{-1}\). In this event every point of \(I_N\) is within \(\varepsilon N^{-1}\) of \(A\), and this allows us to introduce a factor \(\varepsilon^{\beta-1}\) into the previous estimate of \(|g(I_N)|\), still preserving the number of intervals \(I_N\).

(ii) \(m(I_N \cap A) \leq (1-\varepsilon)N^{-1}\). Let us write \(\nu_N\) for the number of the intervals, and \(\nu_N'\) for the number treated in (i). Then \(m(A) \ll N^{-1}\nu_N' + (1-\varepsilon)N^{-1}\nu_N\). But because \(A\) is closed, \(\nu_N + \nu_N' = Nm(A) + o(N)\),

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hence \( \nu_N = o(N) \). Lemma 2 is an easy consequence of this fact and the estimate given in (i).

To obtain the result on the \( 1/\beta \)-measure of \( f(Z) \) we select \( A = Z' \) and note that \( Z \sim Z' \) is countable. It is worth remarking that if \( Df \geq 0 \) and \( m(Z) = 0 \) then \( f \) is strictly monotone and \( f(Z \sim Z') \) is the set of isolated points of \( f(Z) \).

2. In this section we suppose that \( F \) is a set described in Theorem 2. Let \( g \) be absolutely continuous on \( [a, b] \supseteq F \), and linear on each contiguous interval \( J \), with derivative \( |J|^{1/\beta - 1} \). The mapping inverse to \( g \), say \( h \), is increasing and continuous because \( Dg > 0 \) almost everywhere. But \( h \) is also absolutely continuous because it maps each (Lebesgue) null set onto a null set. Thus \( F \) is subject to the previous lemma, since \( |g(J)| = |J|^{1/\beta} \) and \( m(g(F)) = 0 \).

In the proof of Theorem 2 we keep the function \( h \), but regard it solely as a mapping of \( g(F) \) onto \( F \). We now extend \( h \) to a mapping of class \( C^\infty \). Let \( \chi \) be a function in \( C^\infty[0, 1] \),

\[
\chi(0) = 0, \quad \chi(1) = 1, \quad D\chi > 0 \text{ on } (0, 1), \quad D^k\chi(0) = D^k\chi(1) = 0, \quad 1 \leq k < \infty.
\]

On each interval \( (t_1, t_2) \) contiguous to \( g(F) \) we define

\[
f(s) = h(t_1) + (t_2 - t_1)^\beta \chi(|s - t_1|/(t_2 - t_1)), \quad t_1 < s < t_2.
\]

Then \( f(t_1+) = h(t_1), \ f(t_2-) = h(t_1) + (t_2 - t_1)^\beta = h(t_2) \). When \( 1 \leq k < \infty \),

\[
D^k f(s) = (t_2 - t_1)^{\beta - k} D^k \chi(|s - t_1|/(t_2 - t_1)).
\]

In particular the \( k \)th derivative of \( f \), on the complement of \( g(F) \), is uniformly bounded for \( 1 \leq k \leq \beta \).

Now \( f \) is absolutely continuous, for it is monotone-increasing and continuous, and preserves null sets. Hence its derivative is given by \( Df \) (extended to all of \( h([a, b]) \)). Also, the functions \( Df, \ldots, D^{n-1}f \) are continuous on \( h[a, b] \), vanish on \( h(F) \), and have uniformly bounded derivatives on the complement of \( h(F) \). It follows that each is the derivative of its predecessor; for the same reasons \( D^nf \) is the derivative of \( D^{n-1}f \), and \( f \) is \( n \) times continuously differentiable. From the formula for \( D^nf \), it vanishes continuously on \( h(F) \), and when \( \beta < n + 1 \), \( D^nf \) satisfies a Lipschitz condition of order \( \beta - n \), on the contiguous intervals. From these facts the Lipschitz condition for all of \( h([a, b]) \) is easily deduced.

To improve this result for \( \beta = n + 1 \), we proceed as follows. Writing \( l_1 \geq l_2 \geq \cdots \geq l_n \geq \cdots \) for the lengths of intervals \( I_n \) complementary to \( F \), we find numbers \( 1 < c_1 < c_2 < \cdots < c_n \to +\infty \) such that
\[ \sum (c_n l_n)^{1/\beta} < \infty. \]

We then modify the function \( g \), so that \( I_n \) is mapped onto an interval of length \( (c_n l_n)^{1/\beta} \). The function \( h \) inverse to \( g \) is also modified and so ultimately is the function \( f \) (constructed with the aid of the auxiliary mapping \( \chi \)). We consider in detail this function, \( \tilde{f} \).

Writing \( (t_1, t_2) \) for the transform by \( g \) of the interval \( I_n \), we have

\[
\tilde{f}(s) = h(t_1) + \frac{c_n}{c_n} (t_2 - t_1)^{\beta} \chi( |s - t_2| / (t_2 - t_1)), \quad t_1 < s < t_2,
\]

\[
D^{n+1} \tilde{f}(s) = c_n D^{n+1} \chi( |s - t_2| / (t_2 - t_1)).
\]

Since the factor \( c^{-1} \) converges to 0 with the length of the interval \( (t_1, t_2) \), \( \tilde{f} \) belongs to the conventional class \( C^{n+1} \).

3. In this section we show that the vanishing of the \( 1/\beta \)-measure of \( f(Z) \) cannot be strengthened very much. Let \( q \) be a function on \( (0, \infty) \) such that \( q(t) \) and \( t^{1/\beta}/q(t) \) are increasing, \( 2^{m} q(2^{-m}) < \infty \).

**Theorem 3.** There exists a function \( f \) satisfying the conditions of Theorem 1, for which \( f = F(Z) \) has positive Hausdorff measure with respect to the function \( \phi(t) = t^{1/\beta}/q(t) \).

Choosing \( q(t) = \log^2(2^{-t}) \) for small \( t \), we find that \( F(Z) \) can have dimension \( 1/\beta \).

**Proof.** Without loss of generality we can suppose \( \sum q(2^{-m}) < 1. \) In each dyadic interval \( [k2^{-m}, (k+1)2^{-m}] \subseteq [0, 1] \) we construct an interval centered at \( (k+\frac{1}{2})2^{-m} \), of length \( 2^{-m}q(2^{-m}) \). We remove all intervals defined for \( m = 1 \), then all intervals defined for \( m = 2 \) save those intersecting an interval already removed, and so on. The disjoint intervals selected form an open set \( W \) of measure \( m(W) < 1. \) Let \( Df = 0 \) on \( Z = [0, 1] \sim W \), and on an interval \( I \) of \( W \), let \( Df = |I|^{-\delta}. \) Then \( f(Z) \) is a set \( F \), since the contiguous intervals have lengths \( |I|^{-\delta} \) corresponding to the components \( I \) of \( W \).

Observe next that if \( s_1 < s_2 \) and \( s_1, s_2 \in Z \), the \( f(s_2) - f(s_1) \gg (s_2 - s_1)^{\beta}q(2^{-m}) \). Indeed \( (s_1, s_2) \) contains a dyadic interval \( [k2^{-m}, (k+1)2^{-m}] \), with \( 2^{-m} = \frac{1}{4}(s_2 - s_1). \) The interval constructed in \( [k2^{-m}, (k+1)2^{-m}] \) either belongs to \( W \), or intersects a larger interval contained in \( W \), of length \( \geq 2^{-m}q(2^{-m}) \). In any case an interval of that length belongs entirely to \( W \sim (s_1, s_2) \), whence the lower bound on \( f(s_2) - f(s_1) \).

Let \( \mu \) be the measure of Borel sets \( E \) defined by

\[
\mu(E) = m(Z \cap f^{-1}(E)), \quad \mu(f(Z)) = m(Z) > 0.
\]

The proof will be completed by showing that \( \mu(I) \ll \phi(|I|) \) for all
intervals $I$. Now $I$ contains a subinterval $I_0$ with end points in $f(Z)$, such that $\mu(\text{int}(I_0)) = \mu(I)$, and of course $\phi(\text{int}(I_0)) \leq \phi(\text{int}(I))$.

Let $I_0 = f(J)$, for an interval $J$ contiguous to $W$. Then $\mu(I_0) \leq |J|$, while $|I_0| \gg |J|^\beta(q(\text{int}(J))$. Thus

$$\mu(I_0) \ll |I_0|^{1/\beta}/q(\text{int}(J)) \ll |I_0|^{1/\beta}/q(|I_0|),$$

because $|I_0| \ll |J|$.

Related questions in Euclidean space have been treated by Sard in [2].

References


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