AN INEQUALITY FOR THE Riemann-Stieltjes Integral

Richard Darst and Harry Pollard

Abstract. Let g and h be real valued and continuous on the interval [a, b], and suppose that the variation, V[h], of h on [a, b] is finite. By completely elementary methods, it is shown that V[h] · sup_{a \leq a < \beta \leq b} (g(\beta) - g(a)) is an upper bound for \( \int_a^b (h - \inf h) dg \).

Several writers have recently obtained upper bounds for integrals of the form \( \int_a^b h dg \), where h is of bounded variation on the interval [a, b] and g is continuous there ([1], [2], [3, p. 573], [4]). It is our purpose to establish the following extension by completely elementary methods.

Theorem. If h is real and of bounded variation on the interval [a, b] and g is real and continuous there, then

\[
\int_a^b h dg \leq (\inf h)(g(b) - g(a)) + S[a, b] V[h],
\]

where \( V[h] \) is the total variation of h, and

\[
S[a, b] = \sup_{a \leq a < \beta \leq b} \int_a^\beta dg.
\]

We observe first that it is enough to prove the inequality in the case \( \inf h = 0 \), when it becomes

\[
(*) \quad \int_a^b h dg \leq S[a, b] V[h].
\]

For the general case can be obtained from (*) by replacing h in it by \( h - \inf h \). Clearly we may also suppose that for some \( \xi \), \( h(\xi) = 0 \). Since

\[
\int_a^b h dg = \int_a^\xi h dg + \int_{\xi}^b h(- x) d[- g(- x)]
\]

and \( \sup_{\alpha \leq a < \beta \leq \xi} ([- g(- \beta)] - [- g(- \alpha)]) = S[\xi, b] \), we need only show...
(L) If $h \geq 0$ and $h(b) = 0$, then

$$\int_a^b hdg \leq S[a, b] V[h].$$

**Proof of (L).** Assume $g(a) = 0$. Let $\phi(t) = \inf_{t \in [a, t]} g(\xi)$ and let $\psi(t) = g(t) - \phi(t) = \sup_{t \in [a, t]} (g(t) - g(\xi)) \leq S[a, t]$.

Then $\phi$ is nonincreasing, $\phi(a) = 0$, and $0 \leq \psi(t) \leq S[a, t]$. Moreover

$$\int_a^b hdg = \int_a^b h d\phi + \int_a^b h d\psi \leq 0 + \int_a^b h d\psi$$

$$= - \int_a^b \psi dh \leq ||\psi||_w V[h] \leq S[a, b] V[h].$$

**References**


**Purdue University, Lafayette, Indiana 47907**