ON MULTIPLIERS AND ORDER-BOUNDED OPERATORS IN $C^*$-ALGEBRAS

TAGE BAI ANDERSEN

Abstract. It is shown that the selfadjoint multipliers from a sub-$C^*$-algebra to the bigger $C^*$-algebra are exactly the order-bounded operators. As a corollary we get a characterization of the relative commutant of a sub-$C^*$-algebra with unit.

If $A$ is a $C^*$-algebra, we shall denote the selfadjoint part of $A$ by $A_{a}$, the positive part of $A$ by $A^{+}$, and the unit ball of $A$ by $A_{1}$. If $A$ and $B$ are $C^*$-algebras a linear map $T$ from $A$ into $B$ will be called selfadjoint if $T(a^*) = (Ta)^*$ for all $a$ in $A$.

Definition 1. Let $A$ be a sub-$C^*$-algebra of the $C^*$-algebra $B$. A selfadjoint linear map $T$ from $A$ into $B$ will be called order-bounded if there is a nonnegative real number $\lambda$ such that

$$-\lambda a \leq Ta \leq \lambda a, \quad \text{all } a \in A^+.$$ 

Remark 1. If $T$ is order-bounded then $T$ is automatically continuous. In fact, if $a \in A^+$ then $\|Ta\| \leq \lambda \|a\|$. If $a \in A$ then $a = a_1 - a_2 + i(a_3 - a_4)$, where $a_i \in A^+$ and $\|a_i\| \leq \|a\|$ for $i = 1, \ldots, 4$. Then $\|Ta\| \leq 4\lambda \|a\|$.

Definition 2. Let $A$ be a sub-$C^*$-algebra of the $C^*$-algebra $B$. A bounded linear map $T$ from $A$ into $B$ is said to act as a multiplier if

$$T(ab) = (Ta)b = a(Tb), \quad \text{all } a, b \in A.$$ 

(A multiplier is a double centralizer in the sense of [3].)

Theorem 1. Let $A$ be a sub-$C^*$-algebra of the $C^*$-algebra $B$. Let $T$ be a bounded selfadjoint linear map from $A$ into $B$. Then the following conditions are equivalent:

(i) $T$ acts as a multiplier.
(ii) $T$ is order-bounded.

Proof. (i) $\Rightarrow$ (ii). Let $x \in A^+_1$, and let $0 < \alpha < 1$. Since $T$ acts as a multiplier: $x^\alpha(Tx^{1-\alpha}) = Tx = (Tx^{1-\alpha})x^\alpha$, and hence $x^\alpha$ and $Tx^{1-\alpha}$ commute. Let $f$ be a positive linear functional on $B$. Then $(Tx^{1-\alpha})x^\alpha \leq \|Tx^{1-\alpha}\|x^\alpha$ by Gelfand theory and
\[ f(Tx) = f((Tx^{1-\alpha})x^\alpha) \leq f(\|Tx^{1-\alpha}\|x^\alpha) \]
\[ \leq \|T\|f(x^\alpha) \to \|T\|f(x) \quad \text{for} \quad \alpha \to 1. \]

(Since \( x^\alpha \to x \) in norm by Gelfand theory.) Analogously, \(-\|T\|f(x) \leq f(Tx)\), and since the order of \( B \) is completely determined by the positive linear functionals \([2, 2.6.2]\) we get \(-\|T\|x \leq Tx \leq \|T\|x\) and \( T \) is order-bounded.

(ii)\(\Rightarrow\)(i). By replacing \( T \) by \((2\lambda)^{-1}(\lambda I + T)\) we may assume: \(0 \leq Ta \leq a\), for all \( a \in A^+\).

Let \( \pi \) be any faithful representation of \( B \), e.g. the universal representation of \( B \) \([2, 2.7.6]\). By considering \( \pi(A), \pi(B) \) and \( \pi \circ T \circ \pi^{-1} \) we may assume \( A \) and \( B \) to be concrete \( C^* \)-algebras acting on a Hilbert space \( H \) such that \( B \) is nondegenerate.

We notice that if \( \{a_\gamma\}_{\gamma \in G} \) is a net from \( A_1^+ \) tending weakly to \( 0 \) then for all \( \gamma \in G \) and \( \xi \in H \)
\[ 0 \leq (Ta_\gamma\xi, \xi) \leq (a_\gamma\xi, \xi) \]
and hence \( (Ta_\gamma\xi, \xi) \to 0 \). By a result of Kadison \([4, \text{Remark 2.2.3}]\) \( T \) has an extension \( \overline{T} \) to the weak closure \( \overline{A} \) of \( A \) into \( B \) such that \( \overline{T} \) is weakly continuous on the unit ball \( A_1 \) of \( \overline{A} \).

If \( a \in A_1^+ \) there is by Kaplansky's density theorem \([1, \text{Theorem 3, p. 46}]\) a net \( \{a_\gamma\}_{\gamma \in G} \subseteq A_1^+ \) such that \( a_\gamma \to a \) strongly. For each \( \gamma \in G \) and \( \xi \in H \) we get
\[ 0 \leq (Ta_\gamma\xi, \xi) \leq (a_\gamma\xi, \xi) \]
and since \( \overline{T} \) is weakly continuous on \( \overline{A} \), \( 0 \leq (\overline{T}a_\xi, \xi) \leq (a\xi, \xi) \), and \( \overline{T} \) is order-bounded.

Since \( B \) (resp. \( \overline{A} \)) is weakly closed \( \overline{B} \) (resp. \( \overline{A} \)) has a unit \( e_1 \) (resp. \( e_2 \)) \([1, \text{Theorem 2, p. 44}]\).

Since \( B \) is nondegenerate \( e_1 \) is the identity operator on \( H \), while \( e_2 \) is a projection in \( B \), and \( \overline{A} \) is a von Neumann algebra on the Hilbert space \( e_2 H \).

Let \( p \) be a projection in \( \overline{A} \). Then \( 0 \leq \overline{T}p \leq p \) and so for all \( \xi \in H \)
\[ 0 \leq ((\overline{T}p)^{1/2}(e_1 - p)\xi, (\overline{T}p)^{1/2}(e_1 - p)\xi) \]
\[ \leq ((\overline{T}p)(e_1 - p)\xi, (e_1 - p)\xi) \]
\[ \leq (p(e_1 - p)\xi, (e_1 - p)\xi) = 0. \]

This means that \( (\overline{T}p)^{1/2}(e_1 - p) = 0 \) and hence \( (\overline{T}p)(e_1 - p) = 0 \) i.e. \( \overline{T}p = (\overline{T}p) \cdot p \).

Since \( \overline{T}p \) is selfadjoint we also get \( \overline{T}p = p(\overline{T}p) \). Now we get
\begin{align*}
(Te_2)p &= (Tp)p + (Te_2 - p)p \\
&= Tp + (Te_2 - p)(e_2 - p)p \\
&= Tp
\end{align*}

and analogously \( p(Te_2) = Tp \).

Since \( T \) is norm-continuous on \( \overline{A} \), and \( Tp = p(Te_2)p = p(Te_2) \) for all projections \( p \) in \( \overline{A} \) it follows from the spectral theorem that

\[ Ta = (Te_2)a = a(Te_2), \quad \text{for all } a \in \overline{A}, \]

and hence for all \( a \in \overline{A} \). Especially, \( Te_2 \in \mathcal{A}' \).

Now let \( a, b \in A \). Then

\[ T(ab) = T(ab) = (Te_2)(ab) = ((Te_2)(a))b = (Ta)b = (Ta)b \]

and since \( Te_2 \in \mathcal{A}' \) we get analogously \( T(ab) = a(Tb) \), and the theorem is proved.

If \( A \) has a unit we can obtain the following characterization of the relative commutant of \( A \) in \( B \):

**Corollary 1.** Let \( A \) be a sub-C*-algebra of the C*-algebra \( B \), and assume that \( A \) has a unit \( e \). Let \( T \) be a bounded selfadjoint linear operator from \( A \) into \( B \). Then the following conditions are equivalent:

(i) \( T \) is order-bounded.

(ii) \( T \) is multiplication by a selfadjoint element of \( A' \cap B \) (e.g. \( Te \)).

**Proof.** (i) \( \Rightarrow \) (ii). By Theorem 1 \( T \) acts as a multiplier. Then for all \( a \in A : Ta = T(\mathbf{e}a) = (Te)a \) and \( Ta = T(\mathbf{e}a) = a(\mathbf{T}e) \). Hence \( Te \in \mathcal{A}' \cap B \), and the result follows.

(ii) \( \Rightarrow \) (i). This follows either directly from Theorem 1 or by the observation that if \( b \) is a selfadjoint element from \( A' \cap B \) and \( a \in A^+ \) then:

\[ -\|b\|a \leq ba \leq \|b\|a. \]

If we specialize \( A = B \) we get the following well-known result

**Corollary 2.** Let \( A \) be a C*-algebra with unit. Let \( T \) be a bounded selfadjoint linear operator in \( A \). Then \( T \) is order-bounded if and only if \( T \) is multiplication by a selfadjoint central element in \( A \).

**Corollary 3.** Let \( A \) and \( B \) be C*-algebras, \( \Phi \) a *-homomorphism of \( A \) into \( B \), and \( T \) a bounded selfadjoint linear operator from \( A \) into \( B \). If there is a nonnegative real number \( \lambda \) such that

\[ -\lambda \Phi(a) \leq Ta \leq \lambda \Phi(a), \quad \text{all } a \in A^+, \]

then
T(ab) = (Ta)Φ(b) = Φ(a)(Tb), \quad \text{all } a, b \in A.

Moreover, if \( A \) has a unit \( e \) then \( Te \in \Phi(A)' \cap B \) and \( Ta = (Te) \Phi(a) \).
Especially, \( T \) is a \(*\)-homomorphism if and only if \( Te \) is a projection in \( B \cap \Phi(A)' \).

**Proof.** Since the kernel of \( \Phi \) is positively generated \( T \) is 0 on \( \ker(\Phi) \).
Hence the map \( \Psi: \Phi(A) \to B \) is well defined by \( \Psi(\Phi(a)) = Ta, \) all \( a \in A \),
as a linear operator. Since \( T \) is selfadjoint \( \Psi \) is selfadjoint. \( \Phi(A) \) is a sub-\( C^* \)-algebra of \( B \), and since \( \Phi(A^+) = \Phi(A)^+ \), \( \Psi \) is order-bounded.
By Remark 1 \( \Psi \) is continuous. By Theorem 1, \( \Psi \) acts as a multiplier, and hence for all \( a, b \in A \)
\[
T(ab) = \Psi(\Phi(ab)) = \Psi(\Phi(a)\Phi(b)) = \Psi(\Phi(a))\Phi(b) = (Ta)\Phi(b)
\]
and analogously \( T(ab) = \Phi(a)(Tb) \).

Now the second part follows easily from the first part.

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**References**