CHARACTERIZATION OF THE FINITE PARTITION PROPERTY FOR A COLLECTION OF UNIVERSAL SUBCONTINUA

WILLIAM T. TROTTER, JR.

Let $X$ be a Hausdorff space and $A \subset X$ a continuum. $A$ is said to be a Universal Subcontinuum (USC) if $A \cap B$ is connected for every continuum $B \subset X$. Let $\alpha$ be a collection of USC's of a Hausdorff space. Then $\alpha$ is said to have the finite partition property if $\alpha$ has a decomposition into a finite number of subcollections each having the finite intersection property. A result due to W. J. Gray [1] can be easily modified to show that in a Hausdorff space, a collection of USC's has the finite intersection property if every pair has a common point. Other properties of USC's are given in [2] and [3].

**Theorem.** Let $\alpha$ be a collection of USC's of a Hausdorff space. Then the following statements are equivalent.

1. $\alpha$ has the finite partition property.
2. There exist integers $p, q$ with $p \geq q \geq 2$ such that for every $p$ elements of $\alpha$, at least $q$ of them have a common point.
3. $\alpha$ has no infinite pairwise disjoint subcollection.

**Proof.** (1) implies (2) and (2) implies (3) are obvious. Condition (2) is used in [2] to obtain a result which states that the maximal number of subcollections required for the partition is $p - q + 2$. We now prove that (3) implies (1). The proof is by contradiction; we assume that $\alpha$ is a collection of USC's of a Hausdorff space with no infinite pairwise disjoint subcollection, but that $\alpha$ does not have the finite partition property.

Let $I(\alpha) = \{ \beta \subset \alpha \mid \beta$ is pairwise disjoint $\}$. Let $\beta_1, \beta_2 \in I(\alpha)$. We say $\beta_1 \leq \beta_2$ if $\beta_1 \subset \beta_2$. Then it is clear that $\leq$ defines a partial order on $I(\alpha)$. Let $\{ \beta_j \mid j \in J \}$ be a totally ordered subset of $I(\alpha)$. Then define $\beta = \bigcup_{j \in J} \beta_j$. We show $\beta \in I(\alpha)$. Let $H, G \in \beta$, then there exist $j_1, j_2 \in J$ such that $H \in \beta_{j_1}$ and $G \in \beta_{j_2}$. We may assume $\beta_{j_1} \supset \beta_{j_2}$ which implies that $H, G \in \beta_{j_2}$ and that $H \cap G = \emptyset$ if $H \neq G$. Thus $\beta$ is pairwise disjoint and $\beta \in I(\alpha)$. It is clear that $\beta \geq \beta_j$ for every $j \in J$ and hence $\beta$ is an upper bound for the chain. Thus every pairwise disjoint subcollection of $\alpha$ is a subset of some maximal element of $I(\alpha)$.

Let $\alpha = \alpha_0$; we define the following subcollections inductively:

$\beta_i, \alpha_i^1, \alpha_i^2, \beta_i^1, \alpha_i^3, \alpha_{i+1}$.

Received by the editors May 9, 1969.
(1) $\beta_i$ is a maximal element of $I(\alpha_i)$ with $\text{card } \beta_i \geq 2$.

(2) $\alpha_i^1 = \{ H \subseteq \alpha_i \mid \text{there exist } B_1, B_2 \subseteq \beta_i \text{ such that } B_1 \cap B_2 = \emptyset \text{ and } H \cap B_1 \neq \emptyset \neq H \cap B_2 \}$.

(3) $\alpha_i^2 = \{ H \subseteq \alpha_i - \alpha_i^1 \mid \text{there exists } B_1, B_2 \subseteq \beta_i \text{ and } G \subseteq \alpha_i - \alpha_i^1 \text{ such that } B_1 \cap B_2 = \emptyset, H \cap B_1 \neq \emptyset \neq G \cap B_2, \text{ and } H \cap G \neq \emptyset \}$.

(4) $\beta_i^1 = \{ B \subseteq \beta_i \mid \text{there exist } G_B, H_B \subseteq \alpha_i - \alpha_i^1 - \alpha_i^2 \text{ such that } G_B \cap B \neq \emptyset \neq H_B \cap B \}$.

(5) $\alpha_i^3 = \{ H \subseteq \alpha_i - \alpha_i^1 - \alpha_i^2 \mid \text{there exists } B \subseteq \beta_i - \beta_i^1 \text{ such that } H \cap B \neq \emptyset \}$.

(6) $\alpha_{i+1} = \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3$.

We proceed as follows: Since $\alpha_0$ does not have the finite intersection property, there exist $\gamma, \delta \subseteq \alpha_0$ such that $\gamma \cap \delta = \emptyset$. Then there exists a maximal element $\beta_0 \in I(\alpha_0)$ with $\{ \gamma, \delta \} \subseteq \beta_0$. We now show that $\alpha_0^1$ and $\alpha_0^2$ have the finite partition property.

We associate each element of $\alpha_0^1$ with the disjoint pair $B_1, B_2 \subseteq \beta_0$ given in the definition of $\alpha_0^1$ and then show that the collection of all elements associated with the same pair has the finite intersection property. Let $H_1, H_2 \subseteq \alpha_0; B_1, B_2 \subseteq \beta_0; H_1 \cap B_1 \neq \emptyset \neq H_1 \cap B_2; H_2 \cap B_1 \neq \emptyset \neq H_2 \cap B_2; \text{ and } B_1 \cap B_2 = \emptyset$. Suppose $H_1 \cap H_2 = \emptyset$. Then $B_1 \cup H_1 \cup B_2$ and $B_1 \cup H_2 \cup B_2$ are USC's but $(B_1 \cup H_1 \cup B_2) \cap (B_1 \cup H_2 \cup B_2) = B_1 \cup B_2 = B_1 \cup B_2$. The contradiction shows $H_1 \cap H_2 \neq \emptyset$. By hypothesis, card $\beta_0$ is finite and it is clear that $\alpha_0^1$ has the finite partition property. The proof that $\alpha_0^2$ has the finite partition property is similar.

Now suppose that $\beta_0^1 = \emptyset$. Then since $\beta_0$ is maximal, every $H \subseteq \alpha_0$ intersects some $B \subseteq \beta_0$. We then decompose $\alpha_0 - \alpha_0^1 - \alpha_0^2$ into at most card $\beta_0$ subcollections by associating each element with the unique $B \subseteq \beta_0$ which it intersects. Since $\beta_0^1 = \emptyset$, every pair associated with the same $B$ have a common point. Thus $\alpha_0 - \alpha_0^1 - \alpha_0^2$ has the finite partition property and hence so does $\alpha_0$. The contradiction shows $\beta_0^1 \neq \emptyset$.

It is clear that $\alpha_0^3$ has the finite partition property and that $\{ G_B, H_B \mid B \subseteq \beta_0^3 \} \subseteq I(\alpha_0)$. We choose a maximal element $\beta_1 \subseteq I(\alpha_0)$ such that $\{ G_B, H_B \mid B \subseteq \beta_0^1 \} \subseteq \beta_1$. It $\alpha_1$ has the finite partition property, then so does $\alpha_0 = \alpha_1 \cup \alpha_0^1 \cup \alpha_0^2 \cup \alpha_0^3$. Therefore we assume that $\alpha_1$ does not have the finite partition property, and it is clear that this argument can be repeated inductively.

We now show that the sequence $\{ \beta_i \}$ satisfies the following property:

(p) Suppose $i, j,$ and $k$ are integers such that $i \geq j$ and $i \geq k$. Let $B_i \subseteq \beta_i, B_j \subseteq \beta_j,$ and $B_k \subseteq \beta_k$. Then $B_i \cap B_j \neq \emptyset \neq B_i \cap B_k$ implies $B_j \cap B_k \neq \emptyset$.

Assume $B_j \cap B_k = \emptyset$. If $i = j$, then $B_i = B_j$ and $B_j \cap B_k \neq \emptyset$. Thus
$i \neq j$ and similarly, we have $i \neq k$. If $j = k$, then $B_j \cap B_k = \emptyset$ implies $B_i \in \alpha^1$. Thus $j \neq k$ and we may assume $i > j > k$. Since $B_j \in \alpha_j \subset \alpha_k$ and $\beta_k$ is a maximal element of $I(\alpha_k)$, there exists $H_k \in \beta_k$ such that $B_j \cap H_k \neq \emptyset$ and $H_k \cap B_k = \emptyset$. But this implies $B_j \in \alpha^2_k$. The contradiction shows $B_j \cap B_k \neq \emptyset$.

Let $\gamma_i = \bigcup_{B \in \beta_i} B$. We now show $\cap_{i=0}^n \gamma_i \neq \emptyset$. Since each $\gamma_i$ is compact, it suffices to show $\cap_{k=0}^n \gamma_i \neq \emptyset$ in every $k \geq 0$. Let $B_k \in \beta_k$. Then in every $i \leq k$, we have $B_k \in \alpha_k \subset \alpha_i$ and since $\beta_i$ is a maximal element of $I(\alpha_i)$, there exists $B_i \in \beta_i$ such that $B_i \cap B_k \neq \emptyset$. But property (p) obviously implies $\cap_{i=0}^n B_i \neq \emptyset$ and thus $\cap_{i=0}^n \gamma_i \neq \emptyset$.

Let $x \in \cap_{i=0}^n \gamma_i$. Then for every $i \geq 0$, there exists $D_i \in \beta_i$ such that $x \in D_i$. We now define a collection of USC's $\{E_i | i \geq 0\}$ as follows:

(i) Since card $\beta_0 \geq 2$, there exists $E_0 \in \beta_0$ such that $E_0 \cap D_0 = \emptyset$.

(ii) For every $i \geq 0$, $D_{i+1} \in \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3$ and $D_{i+1} \cap D_i \neq \emptyset$ imply $D_i \in \beta_i$. Therefore, there exists $E_{i+1} \in \beta_{i+1}$ such that $E_{i+1} \cap D_i \neq \emptyset$ but $E_{i+1} \cap D_{i+1} = \emptyset$.

We now show that the infinite collection $\{E_i | i \geq 0\}$ is pairwise disjoint. The proof is by induction. We have $E_0 \cap D_0 = \emptyset$, $E_1 \cap D_0 \neq \emptyset$, $D_1 \cap D_0 \neq \emptyset$, and $E_1 \cap D_1 = \emptyset$. Therefore $\{E_0, E_1, D_1\}$ is pairwise disjoint. Now assume $\{E_0, E_1, \cdots, E_k, D_k\}$ is pairwise disjoint. Then since $D_{k+1} \cap D_k \neq \emptyset$, $E_{k+1} \cap D_k \neq \emptyset$, and $E_{k+1} \cap D_{k+1} = \emptyset$, property (p) implies $\{E_0, E_1, \cdots, E_k, E_{k+1}, D_{k+1}\}$ is pairwise disjoint. By induction, $\{E_i | i \geq 0\}$ is an infinite pairwise disjoint subcollection of $\alpha$. This contradiction completes the proof.

References


University of Alabama, University, Alabama 35486