CHARACTERIZATION OF THE FINITE PARTITION PROPERTY FOR A COLLECTION OF UNIVERSAL SUBCONTINUA

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Let $X$ be a Hausdorff space and $A \subset X$ a continuum. $A$ is said to be a Universal Subcontinuum (USC) if $A \cap B$ is connected for every continuum $B \subset X$. Let $\alpha$ be a collection of USC's of a Hausdorff space. Then $\alpha$ is said to have the finite partition property if $\alpha$ has a decomposition into a finite number of subcollections each having the finite intersection property. A result due to W. J. Gray [1] can be easily modified to show that in a Hausdorff space, a collection of USC's has the finite intersection property if every pair has a common point. Other properties of USC's are given in [2] and [3].

Theorem. Let $\alpha$ be a collection of USC's of a Hausdorff space. Then the following statements are equivalent.

1. $\alpha$ has the finite partition property.
2. There exist integers $p, q$ with $p \geq q \geq 2$ such that for every $p$ elements of $\alpha$, at least $q$ of them have a common point.
3. $\alpha$ has no infinite pairwise disjoint subcollection.

Proof. (1) implies (2) and (2) implies (3) are obvious. Condition (2) is used in [2] to obtain a result which states that the maximal number of subcollections required for the partition is $p - q + 2$. We now prove that (3) implies (1). The proof is by contradiction; we assume that $\alpha$ is a collection of USC's of a Hausdorff space with no infinite pairwise disjoint subcollection, but that $\alpha$ does not have the finite partition property.

Let $I(\alpha) = \{ \beta \subset \alpha \mid \beta \text{ is pairwise disjoint} \}$. Let $\beta_1, \beta_2 \in I(\alpha)$. We say $\beta_1 \preceq \beta_2$ if $\beta_1 \subset \beta_2$. Then it is clear that $\preceq$ defines a partial order on $I(\alpha)$. Let $\{ \beta_j \mid j \in J \}$ be a totally ordered subset of $I(\alpha)$. Then define $\beta = \bigcup_{j \in J} \beta_j$. We show $\beta \in I(\alpha)$. Let $H, G \in \beta$, then there exist $j_1, j_2 \in J$ such that $H \in \beta_{j_1}$ and $G \in \beta_{j_2}$. We may assume $\beta_{j_1} \subset \beta_{j_2}$ which implies that $H, G \in \beta_{j_2}$ and that $H \cap G = \emptyset$ if $H \neq G$. Thus $\beta$ is pairwise disjoint and $\beta \in I(\alpha)$. It is clear that $\beta \succeq \beta_j$ for every $j \in J$ and hence $\beta$ is an upper bound for the chain. Thus every pairwise disjoint subcollection of $\alpha$ is a subset of some maximal element of $I(\alpha)$.

Let $\alpha = \alpha_0$; we define the following subcollections inductively: $\beta_i, \alpha_i^1, \alpha_i^2, \beta_i^1, \alpha_i^3, \alpha_{i+1}$.
(1) $\beta_i$ is a maximal element of $I(\alpha_i)$ with $\text{card } \beta_i \geq 2$.

(2) $\alpha_i^1 = \{ H \in \alpha_i | \text{there exist } B_1, B_2 \in \beta_i \text{ such that } B_1 \cap B_2 = \emptyset \text{ and } H \cap B_1 \neq \emptyset \neq H \cap B_2 \}$.

(3) $\alpha_i^2 = \{ H \in \alpha_i - \alpha_i^1 | \text{there exists } B_1, B_2 \in \beta_i \text{ and } G \in \alpha_i - \alpha_i^1 \text{ such that } B_1 \cap B_2 = \emptyset, H \cap B_1 \neq \emptyset \neq G \cap B_2, \text{ and } H \cap G \neq \emptyset \}.$

(4) $\beta_i^1 = \{ B \in \beta_i | \text{there exist } G_B, H_B \in \alpha_i - \alpha_i^1 - \alpha_i^2 \text{ such that } G_B \cap B \neq \emptyset \neq H_B \cap B \}$.

(5) $\alpha_i^3 = \{ H \in \alpha_i - \alpha_i^1 - \alpha_i^2 | \text{there exists } B \in \beta_i - \beta_i^1 \text{ such that } H \cap B \neq \emptyset \}.$

(6) $\alpha_{i+1} = \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3$.

We proceed as follows: Since $\alpha_0$ does not have the finite intersection property, there exist $H, G \in \alpha_0$ such that $H \cap G = \emptyset$. Then there exists a maximal element $\beta_0 \in I(\alpha_0)$ with $\{ H, G \} \leq \beta_0$. We now show that $\alpha_0^1$ and $\alpha_0^2$ have the finite partition property.

We associate each element of $\alpha_0^1$ with the disjoint pair $B_1, B_2 \in \beta_0$ given in the definition of $\alpha_0^1$ and then show that the collection of all elements associated with the same pair has the finite intersection property. Let $H_1, H_2 \in \alpha_0; B_1, B_2 \in \beta_0; H_1 \cap B_1 \neq \emptyset \neq H_1 \cap B_2; H_2 \cap B_1 \neq \emptyset \neq H_2 \cap B_2;$ and $B_1 \cap B_2 = \emptyset$. Suppose $H_1 \cap H_2 = \emptyset$. Then $B_1 \cup H_1 \cup B_2$ and $B_1 \cup H_2 \cup B_2$ are USC's but $(B_1 \cup H_1 \cup B_2) \cap (B_1 \cup H_2 \cup B_2) = B_1 \cup B_2 = B_1 \setminus B_2$. The contradiction shows $H_1 \cap H_2 \neq \emptyset$. By hypothesis, card $\beta_0$ is finite and it is clear that $\alpha_0^1$ has the finite partition property. The proof that $\alpha_0^2$ has the finite partition property is similar.

Now suppose that $\beta_0^1 = \emptyset$. Then since $\beta_0$ is maximal, every $H \in \alpha_0$ intersects some $B \in \beta_0$. We then decompose $\alpha_0 - \alpha_0^1 - \alpha_0^2$ into at most card $\beta_0$ subcollections by associating each element with the unique $B \in \beta_0$ which it intersects. Since $\beta_0^1 = \emptyset$, every pair associated with the same $B$ have a common point. Thus $\alpha_0 - \alpha_0^1 - \alpha_0^2$ has the finite partition property and hence so does $\alpha_0$. The contradiction shows $\beta_0^1 \neq \emptyset$.

It is clear that $\alpha_0^3$ has the finite partition property and that $\{ G_B, H_B \mid B \in \beta_0^3 \} \subseteq I(\alpha_0)$. We choose a maximal element $\beta_1 \in I(\alpha_0)$ such that $\{ G_B, H_B \mid B \in \beta_0^3 \} \leq \beta_1$. It $\alpha_1$ has the finite partition property, then so does $\alpha_0 = \alpha_1 \cup \alpha_0^1 \cup \alpha_0^2 \cup \alpha_0^3$. Therefore we assume that $\alpha_1$ does not have the finite partition property, and it is clear that this argument can be repeated inductively.

We now show that the sequence $\{ \beta_i \}$ satisfies the following property:

(p) Suppose $i, j, k$ are integers such that $i \geq j$ and $i \geq k$. Let $B_i \in \beta_i, B_j \in \beta_j$, and $B_k \in \beta_k$. Then $B_i \cap B_j \neq \emptyset \neq B_i \cap B_k$ implies $B_j \cap B_k \neq \emptyset$.

Assume $B_j \cap B_k = \emptyset$. If $i = j$, then $B_i = B_j$ and $B_j \cap B_k \neq \emptyset$. Thus
Let \( i \neq j \) and similarly, we have \( i \neq k \). If \( j = k \), then \( B_j \cap B_k = \emptyset \) implies \( B_i \subseteq \alpha_i^1 \). Thus \( j \neq k \) and we may assume \( i > j > k \). Since \( B_j \subseteq \alpha_j \subseteq \alpha_k \) and \( \beta_k \) is a maximal element of \( I(\alpha_k) \), there exists \( H_k \in \beta_k \) such that \( B_j \cap H_k \neq \emptyset \) and \( H_k \cap B_k = \emptyset \). But this implies \( B_j \subseteq \alpha_k^2 \). The contradiction shows \( B_j \cap B_k \neq \emptyset \).

Let \( \gamma_i = \bigcup_{B \in \beta_i} B \). We now show \( \bigcap_{i=0}^n \gamma_i \neq \emptyset \). Since each \( \gamma_i \) is compact, it suffices to show \( \bigcap_{i=0}^k \gamma_i \neq \emptyset \) in every \( k \geq 0 \). Let \( B_k \in \beta_k \). Then in every \( i \leq k \), we have \( B_k \subseteq \alpha_k \subseteq \alpha_i \) and since \( \beta_i \) is a maximal element of \( I(\alpha_i) \), there exists \( B_i \in \beta_i \) such that \( B_k \cap B_i \neq \emptyset \). But property (p) obviously implies \( \bigcap_{i=0}^n B_i \neq \emptyset \) and thus \( \bigcap_{i=0}^n \gamma_i \neq \emptyset \).

Let \( x \in \bigcap_{i=0}^n \gamma_i \). Then for every \( i \geq 0 \), there exists \( D_i \in \beta_i \) such that \( x \in D_i \). We now define a collection of USC's \( \{ E_i \mid i \geq 0 \} \) as follows:

(i) Since \( \text{card } \beta_0 \geq 2 \), there exists \( E_0 \in \beta_0 \) such that \( E_0 \cap D_0 = \emptyset \).

(ii) For every \( i \geq 0 \), \( D_{i+1} \subseteq \alpha_i - \alpha_i^1 - \alpha_i^2 - \alpha_i^3 \) and \( D_{i+1} \cap D_i \neq \emptyset \) imply \( D_i \subseteq \beta_i^1 \). Therefore, there exists \( E_{i+1} \in \beta_i^1 \) such that \( E_{i+1} \cap D_i \neq \emptyset \) but \( E_{i+1} \cap D_{i+1} = \emptyset \).

We now show that the infinite collection \( \{ E_i \mid i \geq 0 \} \) is pairwise disjoint. The proof is by induction. We have \( E_0 \cap D_0 = \emptyset, E_1 \cap D_0 \neq \emptyset, D_1 \cap D_0 \neq \emptyset, \) and \( E_1 \cap D_1 = \emptyset \). Therefore \( \{ E_0, E_1, D_1 \} \) is pairwise disjoint. Now assume \( \{ E_0, E_1, \ldots, E_k, D_k \} \) is pairwise disjoint. Then since \( D_{k+1} \cap D_k \neq \emptyset, E_{k+1} \cap D_k \neq \emptyset, \) and \( E_{k+1} \cap D_{k+1} = \emptyset \), property (p) implies \( \{ E_0, E_1, \ldots, E_k, E_{k+1}, D_{k+1} \} \) is pairwise disjoint. By induction, \( \{ E_i \mid i \geq 0 \} \) is an infinite pairwise disjoint subcollection of \( \alpha \). This contradiction completes the proof.

References