ON THE 1-1 SUM OF TWO BOREL SETS

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Abstract. It is shown that there exists a Borel subset $B$ of the
real line and a homeomorphism $\phi$ of the real line such that the
set $\{x - \phi(x) ; x \in B\}$ is not a Borel set.

Since the range, $\phi(B)$, of $\phi$ is a Borel set, this result complements a
recent result [1] of Paul Erdős and Arthur H. Stone who showed that
there exist Borel subsets $C, D$ of the real line whose sum $C + D$
$= \{x + y ; x \in C, y \in D\}$ is not Borel.

We begin a verification of our assertion by recalling that there exist
CBV (continuous, bounded variation) functions $h$ on $I = [0, 1]$ to $I$
such that $\text{card}\{y ; \text{card} [h^{-1}(y)] > \aleph_0\} > \aleph_0$.

For the reader's sake, Arthur H. Stone very kindly contributed the
following direct and elementary construction of a suitable $h$. Define
$h$ first on the usual Cantor ternary set $C$ by

$$h\left(\sum_{n=1}^{\infty} a_n \cdot 3^{-n}\right) = \sum_{n=1}^{\infty} a_{2n} \cdot 9^{-n}$$

(where each of $a_1, a_2, \cdots$ is 0 or 2), and extend $h$ to $I$ by making it
linear on each complementary interval. An elementary calculation
then shows

$$|h(x) - h(y)| \leq 3|x - y| \quad \text{for all } x, y \in I,$$

so that $h$ is CBV; and clearly $\text{card} [h^{-1}(y)] = c$ for each of the $c$
numbers $y$ in $h(C)$. Hence [2], there exists a Borel set $A$ such that $h(A)$
is not Borel. Recall that $h$ can be represented as the restriction to $I$
of the difference, $h_1 - h_2$, of two homeomorphisms. Let $B = h_1(A)$ and
let $\phi = h_2 \cdot h_1^{-1}$.

References


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