CONCERNING PRODUCT INTEGRALS AND EXPONENTIALS

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Abstract. Suppose $S$ is a linearly ordered set, $N$ is the set of real numbers, $G$ is a function from $S \times S$ to $N$, and all integrals are of the subdivision-refinement type. We show that if $\int_a^b G^2 = 0$ and either integral exists, then the other exists and $\alpha \prod (1 + G) = \exp \int_a^b G$. We also show that the bounded variation of $G$ is neither necessary nor sufficient for $\int_a^b G^2$ to be zero.

B. W. Helton, J. S. MacNerney, and H. S. Wall have established various relationships between integral equations, sum integrals, and product integrals. This paper establishes a relationship between exponentials, sum integrals, and product integrals which may be used to evaluate certain product integrals or sum integrals. Integrals used are of the subdivision-refinement type and complete definitions of these and other terms and symbols used in this paper may be found in [1] or [2]. Suppose $S$ is a linearly ordered set [2] and $N$ is the set of real numbers. All functions considered will be functions from $S \times S$ to $N$ unless otherwise noted. In [1, Theorem 3.4] it is shown that for functions of bounded variation from $S \times S$ to $N$ the following two statements are equivalent: (1) $\int_a^b G$ exists and (2) $\alpha \prod (1 + G)$ exists. Under the hypothesis that $\int_a^b G^2 = 0$, we show that the following two statements are equivalent for functions from $S \times S$ to $N$: (1) $\int_a^b G$ exists and (2) $\alpha \prod (1 + G)$ exists and is not zero. It is also noted that neither of the following two statements is a consequence of the other.

1) $\int_a^b G^2 = 0$ and (2) $G$ is of bounded variation on $[a, b]$.

Theorem 0. If $\alpha \prod (1 + G)$ exists and is not zero then if $\epsilon > 0$ there is a subdivision $D$ of $\{a, b\}$ such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of $D$, then

$$\left| \log \frac{\alpha \prod (1 + G)}{\prod (1 + G_i)} \right| < \epsilon.$$  

The proof of this theorem is omitted.

Theorem 1. Neither of the following statements is a consequence of the other:

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(1) \( \int_a^b G^2 = 0 \).

(2) \( G \) is of bounded variation.

**Indication of proof.** Let \( G \) be the function such that for each \( 0 \leq x \leq 1, 0 \leq y \leq 1 \),

\[
G(x, y) = x, \quad x = 1/n, \quad n \text{ an integer, and } |x - y| \geq 1/n - 1/(n + 1),
\]

\( = 0 \), otherwise.

\( \int_0^1 G^2 = 0 \) but \( G \) is not of bounded variation on \([0, 1]\) and \( \int_0^1 G \) does not exist. Hence (2) is not a consequence of (1).

Let \( H \) be the function such that for each \( 0 \leq x \leq 1, 0 \leq y \leq 1 \),

\[
H(x, y) = 1, \quad x = 0, \quad y > x,
\]

\( = 0 \), otherwise.

\( V_1^0 H = 1 \) but \( \int_0^1 H^2 = 1 \). Hence (1) does not follow from (2).

The following theorem may be found in \([2, \text{p. 151}]\) and may be established by induction.

**Theorem 2.** If \( n \) is an integer greater than 1 and each of \( \{A_i\}_{i=1}^n \) and \( \{B_i\}_{i=1}^n \) is a sequence of numbers, then

\[
\prod_{i=1}^n A_i - \prod_{i=1}^n B_i = \sum_{i=1}^n \left( \prod_{j=1}^{i-1} B_j \right) (A_i - B_i) \left( \prod_{k=i+1}^n A_k \right).
\]

**Theorem 3.** If \( \int_a^b G^2 \neq 0 \), then the following two statements are equivalent: (1) \( \int_a^b G \) exists.

(2) \( \int_a^b (1 + G) \) exists and is not zero. Furthermore, if either (1) or (2) is true, then \( \int_a^b G = \log a \prod_{i=1}^b (1 + G) \).

**Proof.** 1. Suppose (1) is true and \( \varepsilon > 0 \). Since \( \int_a^b G^2 = 0 \) and \( \int_a^b G \) exist then there is a subdivision \( D \) of \( \{a, b\} \) such that if \( D' \) is a refinement of \( D \), then there is a number \( k \) such that:

(1) \[
\sum_{D'} G_i^2 < \frac{1}{4}
\]

and hence \( |G_i| < \frac{1}{2} \).

(2) \[
\sum_{D'} G_i^2 < \frac{\varepsilon}{2 \exp \left( \frac{3}{2} + \int_a^b G \right)}
\]

(3) \[
|k| < \frac{\varepsilon}{8 \exp \left( \frac{3}{2} + \int_a^b G \right)}
\]
(4) \(|k| < \frac{1}{2}\), so if \(n > m \geq 0\),

\[
\exp(mk/n) < \exp\left(\frac{1}{2}\right) \quad \text{and} \quad \exp(-k) < \exp\left(\frac{1}{2}\right),
\]

and

\[
\int_a^b G = \sum_{D'} G_i + k.
\]

Let \(D' = \{x_i\}_{i=0}^n\) be a refinement of \(D\).

\[
\sum_{i=1}^n \left| \exp\left(G_i + \frac{k}{n}\right) - G_i - 1 \right|
\]

\[
= \sum_{i=1}^n \left| -1 - G_i + \sum_{j=0}^\infty \frac{(G_i + k/n)^j}{j!} \right|
\]

\[
\leq \sum_{i=1}^n \left| \frac{k}{n} \right| + \sum_{i=1}^n \left| \sum_{j=2}^\infty \frac{(G_i + k/n)^j}{j!} \right|
\]

\[
\leq \left| \frac{k}{n} \right| + \sum_{i=1}^n (G_i + k/n)^2 \cdot \left( \sum_{j=2}^\infty \frac{1}{j!} \right)
\]

\[
< \left| \frac{k}{n} \right| + \sum_{i=1}^n (G_i + k/n)^2.
\]

\[
\leq \left| \frac{k}{n} \right| + \frac{\epsilon}{2 \exp\left(\frac{3}{2} + \int_a^b G\right)} + \left| \frac{k}{n} \right| + \left| k \right|
\]

\[
< \frac{7\epsilon}{8 \exp\left(\frac{3}{2} + \int_a^b G\right)}.
\]

Therefore,

\[
\sum_{i=1}^n \left| \exp(G_i + k/n) - G_i - 1 \right|
\]

\[
< \frac{7\epsilon}{8 \exp\left(\frac{3}{2} + \int_a^b G\right)}.
\]

Then,
\[
\left| \prod_{i=1}^{n} (1 + G_i) - \exp \left( \int_{a}^{b} G \right) \right|
\]
\[
= \left| \prod_{i=1}^{n} (1 + G_i) - \prod_{i=1}^{n} \exp(G_i + k/n) \right|
\]
\[
\leq \sum_{i=1}^{n} \left| \prod_{j=1}^{i-1} (1 + G_i) \cdot \exp(G_i + k/n) - 1 - G_i \right| \cdot \prod_{j=i+1}^{n} \exp(G_i + k/n)
\]
\[
\leq \sum_{i=1}^{n} \left| \prod_{j=1}^{i-1} \exp G_i \cdot \prod_{j=i+1}^{n} \exp(G_i + k/n) \cdot \exp(G_i + k/n) - 1 - G_i \right|
\]
\[
= \sum_{i=1}^{n} \exp \left( \sum_{j=1}^{i} G_j + k - G_i - k + ((n - i)/n)k \right)
\]
\[
\cdot \left| \exp(G_i + k/n) - 1 - G_i \right|
\]
\[
< \sum_{i=1}^{n} \exp \left( \int_{a}^{b} G \right) \cdot \exp(\frac{3}{2}) \cdot \exp(\frac{3}{2}) \cdot \exp(\frac{3}{2}) \cdot \left| \exp(G_i + k/n) - 1 - G_i \right|
\]
\[
< \exp \left( \int_{a}^{b} G + \frac{3}{2} \right) \cdot \frac{7\varepsilon}{8 \exp \left( \int_{a}^{b} G + \frac{3}{2} \right)}
\]
\[
< \varepsilon.
\]

Hence, \( \left| \prod_{i=1}^{n} (1 + G_i) - \exp(\int_{a}^{b} G) \right| < \varepsilon \) so that \( \prod_{i=1}^{n} (1 + G) \) exists and is \( \exp(\int_{a}^{b} G) \).

2. Suppose (2) is true and \( \varepsilon > 0 \). Since \( \int_{a}^{b} G^2 = 0 \), \( \prod_{i=1}^{n} (1 + G) \) exists and is not zero, then there exists a subdivision \( D \) of \( \{a, b\} \) such that if \( D' \) is a refinement of \( D \), then

(1) \( \left| G_i \right| < \frac{1}{2} \)

(2) \( \left| \log \frac{\prod_{j=1}^{n} (1 + G)}{\prod_{D'} (1 + G_i)} \right| < \frac{\varepsilon}{2} \)

(3) \( \log(1 + G_i) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} G_i^j}{j} \)

(4) \( M = \sum_{j=1}^{\infty} \frac{\left(\frac{1}{2}\right)^{j-1} j}{j} \geq \sum_{j=2}^{\infty} \frac{|G_i|^{j-2}}{j} \)

(5) \( \sum_{D'} G_i^2 < \frac{\varepsilon}{2M} \).

Let \( D' = \{x_i\}_{i=0}^{n} \) be a refinement of \( D \), then
\[
\left| \log_a \prod_{i=1}^{b} (1 + G) - \sum_{i=1}^{n} G_i \right| \\
\leq \left| \log \prod_{i=1}^{n} (1 + G) - \sum_{i=1}^{n} G_i \right| + \left| \log \prod_{i=1}^{b} (1 + G) \right| \\
< \left| \sum_{i=1}^{n} \left[ \log(1 + G_i) - G_i \right] \right| + \frac{\epsilon}{2} \\
= \left| \sum_{i=1}^{n} \left[ \sum_{j=1}^{\infty} (-1)^{j-1} \frac{G_i^j}{j} - G_i \right] \right| + \frac{\epsilon}{2} \\
= \left| \sum_{i=1}^{n} \sum_{j=2}^{\infty} (-1)^{j-1} \frac{G_i^j}{j} \right| + \frac{\epsilon}{2} \\
= \left| \sum_{i=1}^{n} \left[ G_i^2 \sum_{j=2}^{\infty} (-1)^{j-1} \frac{G_i^{j-2}}{j} \right] \right| + \frac{\epsilon}{2} \\
\leq M \sum_{i=1}^{n} G_i^2 + \frac{\epsilon}{2} < M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon.
\]

Hence,

\[
\left| \log_a \prod_{i=1}^{b} (1 + G) - \sum_{i=1}^{n} G_i \right| < \epsilon
\]

so that \( \int_{a}^{b} G \) exists and is \( \log_a \prod_{i=1}^{b} (1 + G) \).

**Remark.** As noted by the referee, a function \( G \) from \( S \times S \) to \( N \) may have the property that \( \int_{a}^{b} G^2 = 0 \) and \( \int_{a}^{b} G \) exists yet \( G \) fails to be of bounded variation on \( [a, b] \). As an example of such a function we offer the following: Suppose for \( 0 < x < 1 \), \( g(x) = x \sin(\pi/x) \) and \( g(0) = 0 \) and for each \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), \( G(x, y) = g(y) - g(x) \). \( \int_{a}^{b} G^2 = \int_{a}^{b} G = 0 \), but \( \int_{a}^{b} |G| \) does not exist.

**Bibliography**


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