$\mathfrak{F}$-PROJECTIVE OBJECTS

C. E. HALL

ABSTRACT. Previously, projective abelian topological groups were defined based upon the requirement that free abelian topological groups be projective. In the present paper it is observed that a similar definition of projective relative to a functor which has a right adjoint leads to equivalent results in a much more general setting. These results are then specialized to the category of topological groups (with Markov's free functor) for a detailed example.

In a previous paper [1] the notion of projective in the category of abelian topological groups (T_2) was investigated. It is of interest to analyze this concept in the more general category of topological groups (T_2). Both cases, however, are simply examples of the concept phrased in a much more general setting.[1] It is the purpose of this paper to develop the concept in this generality and then to demonstrate how the results specialize to the category of topological groups. Examples other than this are also listed.

I. **Characterization of $\mathfrak{F}$-projective objects.** For a category $\mathcal{C}$ denote $\hom_{\mathcal{C}}(-, -)$ by $\mathcal{C}(-, -)$. Let $\mathcal{S}$ and $\mathcal{G}$ be categories and suppose that the functor $F:\mathcal{S}\to\mathcal{G}$ is left adjoint [3] to the functor $G:\mathcal{G}\to\mathcal{S}$. Let $\mathfrak{F}$ be the class of all morphisms $e:A\to B$ in $\mathcal{G}$ such that the function

$$\mathcal{G}(FX, e):\mathcal{G}(FX, A) \to \mathcal{G}(FX, B)$$

is surjective for all objects X of $\mathcal{S}$.

DEFINITION. An object P of $\mathcal{G}$ is *$\mathfrak{F}$-projective* iff $\mathcal{G}(P, e):\mathcal{G}(P, A) \to\mathcal{G}(P, B)$ is surjective for all $e\in\mathfrak{F}$.

THEOREM 1. *Let $e\in\mathcal{G}(A, B)$. Then $e\in\mathfrak{F}$ iff $Ge\in\mathcal{S}(GA, GB)$ is a retraction (i.e. has a right inverse).*

PROOF. Since F is left adjoint to G there is a natural isomorphism of bifunctors

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[1] The author gratefully acknowledges the contributions of the referee in the development of the concept in the general setting.

$$\eta(X, A):\mathcal{G}(FX, A) = \mathcal{S}(X, GA).$$

The following diagram is then commutative.

$$\begin{tikzcd}
\mathcal{G}(FX, A) \arrow[r, "\eta(X, A)"'] \arrow[d, "\mathcal{G}(FX, e)"'] & \mathcal{S}(X, GA) \arrow[d, "\mathcal{S}(X, Ge)"] \\
\mathcal{G}(FX, B) \arrow[r, "\eta(X, B)"'] & \mathcal{S}(X, GB)
\end{tikzcd}$$

Suppose $e \in \mathfrak{F}$. Then $\mathcal{G}(FX, e)$ is surjective for all X in $\mathcal{S}$ and hence $\mathcal{S}(X, Ge)$ is surjective for all X in $\mathcal{S}$. Let $X = GB$ and let $e' \in \mathcal{S}(GB, GA)$ be such that $\mathcal{S}(GB, Ge)e' = 1_{GB}$. Then $(Ge)e' = 1_{GB}$ and hence Ge is a retraction.

Conversely, suppose Ge is a retraction. Now functors preserve retractions, and $\mathcal{S}(X, -)$ is a functor for all X. Thus all $\mathcal{S}(X, Ge)$ are retractions in the category of sets and hence are surjective. Therefore $\mathcal{G}(FX, e)$ is surjective for all objects X of $\mathcal{S}$ and $e \in \mathfrak{F}$.

THEOREM 2. *The object P of $\mathcal{G}$ is $\mathfrak{F}$-projective iff there is an object X of $\mathcal{S}$ such that P is a retract of FX.*

PROOF. Let $\phi_P = (\eta(GP, P))^{-}(1_{GP}) \in \mathcal{G}(FGP, P)$ and $\psi_{GP} = \eta(GP, FGP)(1_{FGP}) \in \mathcal{S}(GP, GFGP)$. Then $(G\phi_P)\psi_{GP} = 1_{GP}$ and hence $G\phi_P$ is a retraction. Then $\phi_P \in \mathfrak{F}$ by Theorem 1.

Suppose P is $\mathfrak{F}$-projective. Then there is $\phi' \in \mathcal{G}(P, FGP)$ such that $\mathcal{G}(P, \phi_P)\phi' = 1_P$, i.e. $\phi_P\phi' = 1_P$. Hence ϕ_P is a retraction and P is a retract of FGP.

Conversely, suppose $r \in \mathcal{G}(FX, P)$ is a retraction and r' is its right inverse. Let $e: A \to B$ be in $\mathfrak{F}$ and $f \in \mathcal{G}(P, B)$. Since $e \in \mathfrak{F}$, there is $p \in \mathcal{G}(FX, A)$ such that $\mathcal{G}(FX, e)(p) = fr$, i.e. $ep = fr$, by the definition of $\mathfrak{F}$. Then $epr' = frr' = f$, and $\mathcal{G}(P, e)$ is surjective. Thus P is $\mathfrak{F}$-projective.

II. **Examples.** All topological spaces considered as T_2.

1. Let $\mathcal{S}$ be the category of completely regular spaces, and $\mathcal{G}$ be the category of topological groups. Let $G:\mathcal{G} \to \mathcal{S}$ be the forgetful functor. If X is an object in $\mathcal{S}$, let FX denote the free topological group on X as defined by Markov [**4**]. Objects X, X' in $\mathcal{S}$ and $\alpha \in \mathcal{S}(X, X')$ induce $F\alpha \in \mathcal{G}(FX, FX')$. We say $F:\mathcal{S} \to \mathcal{G}$ is the *free functor.* It is clear that $\mathcal{G}(FX, A)$ is naturally isomorphic to $\mathcal{S}(X, GA)$ for all objects X in $\mathcal{S}$ and A in $\mathcal{G}$, so that the functor F is left adjoint to G.

COROLLARY 1. *If $e \in \mathfrak{F}$ then e is an open mapping.*

COROLLARY 2. *A topological group P is $\mathfrak{F}$-projective iff there is a free topological group FX, a continuous injection $\alpha: P \to FX$, and a closed normal subgroup H of FX, all subject to the following conditions:*

(a) $\alpha[P] \cdot H = FX$;

(b) $\alpha[P] \cap H = \{\mathrm{id}\}$;

(c) *if U is open in P then* $\alpha[U] \cdot H$ *is open in FX*;

(d) $\alpha[P]$ *is a closed subset of FX.*

Observe that by Theorem 1, $e \in \mathfrak{F}$ is equivalent to the existence of a continuous cross-section for e. This means that e is open and thus we have Corollary 1. We note that, as a consequence of this corollary, each projective in $\mathcal{G}$ as defined by Hofmann [2] is $\mathfrak{F}$-projective. The conditions (a)–(d) of Corollary 2 are known to be equivalent to P being a retract of FX in $\mathcal{G}$. Thus Corollary 2 is an immediate consequence of Theorem 2.

In connection with this example, it should be observed that results analogous to Theorem 2, Corollary 1, and Corollary 2 in [1] are also valid in the category $\mathcal{G}$.

Many other examples of the results in §I exist. We now list some of these.

2. $\mathcal{S}$ = category of sets, $\mathcal{G}$ = category of [abelian] groups, G = forgetful functor, F = free functor.

3. $\mathcal{S}$ = category of completely regular spaces, $\mathcal{G}$ = category of compact spaces, G = forgetful functor, F = Stone-Čech compactification.

4. $\mathcal{S}$ = category of sets, $\mathcal{G}$ = category of commutative rings (with unit), G = forgetful functor, F = free functor (i.e. FX = polynomial ring $Z[X]$ in the set X of indeterminates).

5. $\mathcal{S} = \mathcal{G}$ = category of sets, $G = \mathcal{S}(S, -)$, $F = S \times -$.

REFERENCES

1. C. Hall, *Projective topological groups*, Proc. Amer. Math. Soc. **18** (1967), 425–431. MR **35** #2994.

2. K. Hofmann, *Zerfällung Topologischer Gruppen*, Math. Z. **84** (1964), 16–37. MR **29** #2321.

3. D. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. **87** (1958), 294–329. MR **24** #A1301.

4. A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 3–64; English transl., Amer. Math. Soc. Transl. (1) **8** (1962), 195–272. MR **7**, 7.

UNIVERSITY OF TEXAS, EL PASO, TEXAS 79999