SHORTER NOTES

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A SUFFICIENT CONDITION THAT THE LIMIT OF A SEQUENCE OF CONTINUOUS FUNCTIONS BE AN EMBEDDING

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Abstract. Suppose $X$ is a metric space, and $Y$ is a complete metric space. In this paper a sufficient condition is given to insure that a sequence of continuous functions from $X$ into $Y$ converge to an embedding from $X$ into $Y$.

In this note we give a sufficient condition that the limit of a sequence of continuous functions be an embedding. This result is a sharpening of one of Bing's results [1].

The difference between Bing's theorem and the one given here is that we remove the requirement that $X$ be compact and relax the requirement of bicontinuity on the $f_n$'s to continuity, in particular the $f_n$'s need not even be 1-1. In the following theorem, by $\rho(f, g)$ we shall mean $\sup_{t \in X} \rho(f(t), g(t))$.

Theorem. Suppose $\{f_n\}$ is a sequence of continuous functions from a metric space $X$ into a complete metric space $Y$ and suppose that $\{t_n\}$ is a sequence of numbers such that $\rho(f_i, f_{i+1}) < t_{i+1}$ for each $i$, and such that $\rho(x_1, x_2) > 1/j$ implies $\rho(f_j(x_1), f_j(x_2)) > 2 \sum_{i=j}^\infty t_i$ for all $x_1, x_2 \in X$. Then $f = \lim_n f_n$ is an embedding.

Proof of Theorem. That $f$ is continuous follows because it is the uniform limit of continuous functions.

Suppose that $y_n \to y_0$ in $f[X] \subseteq Y$. For each $n, n = 0, 1, 2, \cdots$, we chose an $x_n \in X$ such that $f(x_n) = y_n$. Assume that $x_n$ fails to converge to $x_0$. Then, it follows that there is an $\epsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\rho(x_{n_i}, x_0) > \epsilon_0$ for $i = 1, 2, \cdots$. We choose $N$ such that $1/N < \epsilon_0$. Therefore, it follows from our hypothesis that $\rho(f_N(x_{n_i}), f_N(x)) > 2 \sum_{i=N}^\infty \epsilon_i$ for each $i$ since $\rho(x_{n_i}, x_0) \geq \epsilon_0 > 1/N$ for each $i$. Since $f(x_{n_i}) = y_{n_i} \to y_0 = f(x_0)$, then there is a $k$ such that $\rho(f(x_{n_k}), f(x_0)) < \epsilon_N$.

We observe that

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\[ \rho(f_N, f) = \lim_{n} \rho(f_N, f_{N+n}) \leq \lim_{n} \sum_{i=N+1}^{N+n} \epsilon_i = \sum_{i=N+1}^{\infty} \epsilon_i. \]

Therefore,

\[ \rho(f_N(x_{n_k}), f_N(x_0)) \leq \rho(f_N(x_{n_k}), f(x_{n_k})) + \rho(f(x_{n_k}), f(x_0)) + \rho(f(x_0), f_N(x_0)) < 2 \sum_{i=N+1}^{\infty} \epsilon_i + \epsilon_N. \]

But this contradicts the condition \( \rho(f_N(x_{n_k}), f_N(x_0)) > 2 \sum_{i=N}^{\infty} \epsilon_i \), which is guaranteed by our choice of \( \{x_{n_k}\} \). Therefore, we are led to conclude that \( x_n \to x_0 \).

The above paragraph implies the inverse of \( f \) is continuous if the inverse is a single valued function from \( f[X] \) onto \( X \), i.e., if \( f \) is 1-1. However, that \( f \) is 1-1 can also be deduced from the previous paragraph, for if \( f(x) = f(x') = y_0 \), choose \( y_n = y_0 \) for \( n = 1, 2, \ldots \) and choose the sequence \( x_n = x \) for \( n = 1, 2, \ldots \) and \( x_0 = x' \); it follows from the previous paragraph that \( x_n \to x_0 \) which implies \( x = x' \).

Therefore, \( f \) is 1-1 and bicontinuous; hence, an embedding.

**References**


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