ON A CONJECTURE OF E. GRANIRER CONCERNING THE RANGE OF AN INVARIANT MEAN

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ABSTRACT. The purpose of this paper is to prove the following conjecture of E. Granirer: if S is an infinite right cancellation left amenable semigroup then for each left invariant mean ϕ of S, $\{\phi(A):A\subset S\}=[0,1]$.

Let S be a semigroup with discrete topology, $m(S)$ the space of bounded real functions on S with the sup norm. $\phi\in m(S)^*$ is a mean if $\|\phi\|=1$, and $(\phi,f)\geqq 0$ whenever $f\geqq 0$. A mean ϕ is said to be left invariant if $(\phi, l_s f)=(\phi,f)$ for $s\in S$ and $f\in m(S)$, where $l_s f\in m(S)$ is defined by $(l_s f)(s_1)=f(ss_1)$. If $m(S)$ has a left invariant mean then we say S is left amenable.

Each mean ϕ on $m(S)$ can be considered as a finite additive measure on the family of all subsets of S. For $A\subset S$, $\phi(\chi_A)$ will also be denoted by $\phi(A)$. Clearly, if ϕ is a mean then the range of ϕ, $\{\phi(A):A\subset S\}$, is a subset of $[0,1]$. The purpose of this paper is to prove the following.

THEOREM. *Let S be an infinite right cancellation left amenable semigroup. Then the range of each left invariant mean on $m(S)$ is the whole* $[0,1]$ *interval.*

Granirer stated this theorem as a conjecture in [**3**]. There he was able to prove it (in a stronger form) for all cases except when S is a so-called "AB group". An infinite torsion group S is an AB group if (a) S is amenable, (b) each infinite subgroup of S is not locally finite, cf. [**3**].

Each mean ϕ on $m(S)$ corresponds to a unique probability measure μ_ϕ on βS, the Stone-Čech compactification of the discrete set S. The correspondence is characterized by $(\phi,f)=\int_{\beta S} f^- d\mu_\phi$, where $f\in m(S)$ and f^- denotes its continuous extension to βS. If $B\subset S$, B^- will denote the closure of B in βS. Sets of the form B^-, $B\subset S$, are closed-open in βS and they form a topological open basis for βS.

For each $s\in S$, we have a continuous mapping $s^\sim$ of S into βS defined by $s^\sim(s_1)=ss_1$, $s_1\in S$. $s^\sim$ has a unique continuous extension to βS. The extended mapping will also be denoted by $s^\sim$. If S is actually a group then, for each $s\in S$, $s^\sim$ is a homeomorphism from βS onto βS (cf. [**1**, Lemma 2.1]). Moreover, $(s_1^{-1}s_2)^\sim=(s_1^\sim)^{-1}s_2^\sim$ and $e^\sim=$ the iden-

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tity function from βS onto βS, where $s_1, s_2 \in S$ and e is the identity of S.

LEMMA 1. *Let S be an infinite group, $w \in \beta S$, and $s_1, s_2 \in S$, $s_1 \neq s_2$. Then $s_1^{\sim} w \neq s_2^{\sim} w$.*

PROOF. By the remark above, we may assume that $s_1 = e$ and $s_2 = s \neq e$. By Lemmas 1 and 2 of [2], there exist a positive integer k and subsets $A_1, A_2, \cdots, A_k$ of S such that (1) $\bigcup_{i=1}^{k} A_i = S$, (2) $A_i \cap A_j = \emptyset$ if $i \neq j$, and (3) $sA_i \subset A_{i+1}$ if $i \leqq k-1$ and $sA_k \subset A_1$. Note that by (1) $\beta S = \bigcup_{i=1}^{k} A_i^-$. Thus, if $w \in \beta S$ then $w \in A_i^-$ for some i, say, $i = 1$. Then, by (3), $s^{\sim} w \in A_2^-$. Since (2) implies that $A_1^- \cap A_2^- = \emptyset$, we conclude $s^{\sim} w \neq w$.

LEMMA 2. *Let S be an infinite amenable group. If ϕ is a left invariant mean on $m(S)$ then $\mu_\phi(\{w\}) = 0$ for each $w \in \beta S$.*

PROOF. Let $w \in \beta S$. Choose a subset $\{s_1, \cdots, s_n\}$ of S, where $s_i \neq s_j$, if $i \neq j$. Then, by Lemma 1, $s_i^{\sim} w \neq s_j^{\sim} w$ if $i \neq j$. It is clear that we can choose a closed-open neighborhood A^- of w such that $s_i^{\sim} A^- \cap s_j^{\sim} A^- = \emptyset$ if $i \neq j$. Denote the characteristic function of A in S by χ_A. Then

$$\begin{aligned} 1 &= \mu_\phi(\beta S) \geqq \sum_{i=1}^{n} \mu_\phi(s_i^{\sim} A^-) \\ &= \sum_{i=1}^{n} \mu_\phi((s_i A)^-) \qquad \text{(cf. [1, Lemma 2.1])} \\ &= \sum_{i=1}^{n} \phi(s_i A) = \sum_{i=1}^{n} \phi(l_{s_i^{-1}} \chi_A) \\ &= n\phi(A) = n\mu_\phi(A^-) \\ &\geqq n\mu_\phi(\{w\}). \end{aligned}$$

Since n can be arbitrarily big, $\mu_\phi(\{w\}) = 0$.

LEMMA 3. *Let X be an infinite discrete set and ϕ be a mean on $m(X)$ such that $\mu_\phi(\{w\}) = 0$ for each $w \in \beta X$. Then $\{\phi(A) : A \subset X\} = [0, 1]$.*

PROOF. Note first that μ_ϕ is nonatomic. Indeed, if μ_ϕ has atoms then there is a compact atom K. We may cover K by a finite number of open sets $U_1, \cdots, U_n$ with $\mu_\phi(U_i) < \mu_\phi(K)$ for $i = 1, 2, \cdots, n$. Since K is an atom, $\mu_\phi(U_i \cap K) = 0$, and hence, $\mu_\phi(K) = 0$, a contradiction. Consequently, μ_ϕ is nonatomic and by Liapounoff's convexity theorem, cf. [4],[1] $\{\mu_\phi(\Omega) : \Omega$ runs over Borel subsets of $\beta X\} = [0, 1]$.

[1] Many thanks are due to R. Bourgin for referring this paper to us.

Let Ω, a Borel subset of βX, and $\epsilon > 0$ be given. Since μ_ϕ is regular, there exist a closed set Γ and an open set Λ of βX such that $\Lambda \supset \Omega \supset \Gamma$ and $u_\phi(\Lambda \backslash \Gamma) < \epsilon$. Since Γ is compact, we can find a closed-open subset $A^- \subset \beta X$ such that $\Lambda \supset A^- \supset \Gamma$, and hence, $|\mu_\phi(A^-) - \mu_\phi(\Omega)| < \epsilon$. Thus we conclude that $\{\phi(A) : A \subset X\}$ is dense in $[0, 1]$.

Let $\lambda \in (0, 1)$ be given. Choose a sequence (λ_n) in $(0, 1)$ such that $\lambda_{2n-1} > \lambda_{2n+1} > \lambda > \lambda_{2n+2} > \lambda_{2n}$, $n = 1, 2, \cdots$, and $\lim_n \lambda_{2n-1} = \lambda = \lim_n \lambda_{2n}$. Since $\{\phi(A) : A \subset X\}$ is dense in $[0, 1]$, we can choose a set $A_1 \subset X$ such that $\lambda_3 < \phi(A_1) < \lambda_1$. Similarly, since $\{\phi(A) : A \subset A_1\}$ is dense in $[0, \phi(A_1)]$, there exists $A_2 \subset A_1$ such that $\lambda_2 < \phi(A_2) < \lambda_4$. Again, since $\{\phi(A) : A \subset A_1 \backslash A_2\}$ is dense in $[0, \phi(A_1 \backslash A_2)]$, we can choose $B_3 \subset A_1 \backslash A_2$ such that $\lambda_5 < \phi(A_2 \cup B_3) < \lambda_3$. Set $A_3 = A_2 \cup B_3$. Continue this process. We get a sequence of subsets (A_n) of X such that

$$A_{2n-1} \supset A_{2n+1} \supset A_{2n+2} \supset A_{2n}$$

and

$$\lambda_{2n+1} < \phi(A_{2n-1}) < \lambda_{2n-1}, \qquad \lambda_{2n} < \phi(A_{2n}) < \lambda_{2n+2},$$

for $n = 1, 2, \cdots$. Let $A = \bigcap_{n=1}^{\infty} A_{2n-1}$. Then $A_{2n-1} \supset A \supset A_{2n}$, and hence, $\lambda_{2n-1} > \phi(A) > \lambda_{2n}$, $n = 1, 2, \cdots$. Thus $\phi(A) = \lim_n \lambda_n = \lambda$ and the proof is completed.

PROOF OF THE THEOREM. By Theorem 1 and Lemma 4 of [**3**], we may assume that S is an infinite amenable group. The Theorem is then a direct consequence of Lemma 2 and Lemma 3.

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