ON A CONJECTURE OF E. GRANIRER CONCERNING
THE RANGE OF AN INVARIANT MEAN

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Abstract. The purpose of this paper is to prove the following
conjecture of E. Granirer: if $S$ is an infinite right cancellation left
amenable semigroup then for each left invariant mean $\phi$ of $S$,
$\{\phi(A) : A \subseteq S\} = [0, 1]$.

Let $S$ be a semigroup with discrete topology, $m(S)$ the space of
bounded real functions on $S$ with the sup norm. $\phi \in m(S)^*$ is a mean if
$\|\phi\| = 1$, and $(\phi, f) \geq 0$ whenever $f \geq 0$. A mean $\phi$ is said to be left in-
vARIANT if $(\phi, l_sf) = (\phi, f)$ for $s \in S$ and $f \in m(S)$, where $l_sf \in m(S)$ is de-
defined by $(l_sf)(s) = f(ss)$. If $m(S)$ has a left invariant mean then we
say $S$ is left amenable.

Each mean $\phi$ on $m(S)$ can be considered as a finite additive measure
on the family of all subsets of $S$. For $A \subseteq S$, $\phi(\chi_A)$ will also be denoted
by $\phi(A)$. Clearly, if $\phi$ is a mean then the range of $\phi$, $\{\phi(A) : A \subseteq S\}$, is
a subset of $[0, 1]$. The purpose of this paper is to prove the following.

Theorem. Let $S$ be an infinite right cancellation left amenable semi-
group. Then the range of each left invariant mean on $m(S)$ is the whole
$[0, 1]$ interval.

Granirer stated this theorem as a conjecture in [3]. There he was
able to prove it (in a stronger form) for all cases except when $S$ is a so-
called "AB group". An infinite torsion group $S$ is an AB group if (a) $S$
is amenable, (b) each infinite subgroup of $S$ is not locally finite, cf. [3].

Each mean $\phi$ on $m(S)$ corresponds to a unique probability measure
$\mu_\phi$ on $\beta S$, the Stone-Čech compactification of the discrete set $S$. The
correspondence is characterized by $(\phi, f) = \int_{\beta S} f^- d\mu_\phi$, where $f \in m(S)$
and $f^-$ denotes its continuous extension to $\beta S$. If $B \subseteq S$, $B^-$ will denote
the closure of $B$ in $\beta S$. Sets of the form $B^-$, $B \subseteq S$, are closed-open in
$\beta S$ and they form a topological open basis for $\beta S$.

For each $s \in S$, we have a continuous mapping $s^-$ of $S$ into $\beta S$ de-
defined by $s^- s_1 = ss_1$, $s_1 \in S$. $s^-$ has a unique continuous extension to $\beta S$.
The extended mapping will also be denoted by $s^-$. If $S$ is actually a
group then, for each $s \in S$, $s^-$ is a homeomorphism from $\beta S$ onto $\beta S$
(cf. [1, Lemma 2.1]). Moreover, $(s_1 s_2)^- = (s_1^-)^{-1} s_2^-$ and $e^- = \text{the iden-}
tity function from $\beta S$ onto $\beta S$, where $s_1, s_2 \in S$ and $e$ is the identity of $S$.

**Lemma 1.** Let $S$ be an infinite group, $w \in \beta S$, and $s_1, s_2 \in S$, $s_1 \neq s_2$. Then $s_1^{-1} w \neq s_2^{-1} w$.

**Proof.** By the remark above, we may assume that $s_1 = e$ and $s_2 = s \neq e$. By Lemmas 1 and 2 of [2], there exist a positive integer $k$ and subsets $A_1, A_2, \cdots, A_k$ of $S$ such that (1) $\bigcup_{i=1}^{k} A_i = S$, (2) $A_i \cap A_j = \emptyset$ if $i \neq j$, and (3) $sA_i \subseteq A_{i+1}$ if $i \leq k - 1$ and $sA_k \subseteq A_1$. Note that by (1) $\beta S = \bigcup_{i=1}^{k} A_i$. Thus, if $w \in \beta S$ then $w \in A_i$ for some $i$, say, $i = 1$. Then, by (3), $s^{-1} w \in A_i$. Since (2) implies that $A_1 \cap A_i = \emptyset$, we conclude $s^{-1} w \neq w$.

**Lemma 2.** Let $S$ be an infinite amenable group. If $\phi$ is a left invariant mean on $m(S)$ then $\mu_\phi(\{w\}) = 0$ for each $w \in \beta S$.

**Proof.** Let $w \in \beta S$. Choose a subset $\{s_1, \cdots, s_n\}$ of $S$, where $s_i \neq s_j$ if $i \neq j$. Then, by Lemma 1, $s_i^{-1} w \neq s_j^{-1} w$ if $i \neq j$. It is clear that we can choose a closed-open neighborhood $A$ of $w$ such that $s_i^{-1} A \cap s_j^{-1} A = \emptyset$ if $i \neq j$. Denote the characteristic function of $A$ in $S$ by $\chi_A$. Then

$$1 = \mu_\phi(\beta S) \geq \sum_{i=1}^{n} \mu_\phi(s_i^{-1} A)$$

$$= \sum_{i=1}^{n} \mu_\phi((s_i A)^-)$$

$$= \sum_{i=1}^{n} \phi(s_i A) = \sum_{i=1}^{n} \phi(l^{-1}_i \chi_A)$$

$$= n\phi(A) = n\mu_\phi(A^-)$$

$$\geq n\mu_\phi(\{w\}).$$

Since $n$ can be arbitrarily big, $\mu_\phi(\{w\}) = 0$.

**Lemma 3.** Let $X$ be an infinite discrete set and $\phi$ be a mean on $m(X)$ such that $\mu_\phi(\{w\}) = 0$ for each $w \in \beta X$. Then $\{\phi(A) : A \subseteq X\} = [0, 1]$.

**Proof.** Note first that $\mu_\phi$ is nonatomic. Indeed, if $\mu_\phi$ has atoms then there is a compact atom $K$. We may cover $K$ by a finite number of open sets $U_1, \cdots, U_n$ with $\mu_\phi(U_i) < \mu_\phi(K)$ for $i = 1, 2, \cdots, n$. Since $K$ is an atom, $\mu_\phi(U_i \cap K) = 0$, and hence, $\mu_\phi(K) = 0$, a contradiction. Consequently, $\mu_\phi$ is nonatomic and by Liapounoff's convexity theorem, cf. [4], $\mu_\phi(\Omega) : \Omega$ runs over Borel subsets of $\beta X$ = $[0, 1]$.

1 Many thanks are due to R. Bourgin for referring this paper to us.
Let $\Omega$, a Borel subset of $\beta X$, and $\epsilon > 0$ be given. Since $\mu_\phi$ is regular, there exists a closed set $\Gamma$ and an open set $\Lambda$ of $\beta X$ such that $\Lambda \supset \Omega \supset \Gamma$ and $u_\phi(\Lambda \setminus \Gamma) < \epsilon$. Since $\Gamma$ is compact, we can find a closed-open subset $A^- \subset \beta X$ such that $\Lambda \supset A^- \supset \Gamma$, and hence, $|\mu_\phi(A^-) - \mu_\phi(\Omega)| < \epsilon$. Thus we conclude that $\{\phi(A) : A \subset X\}$ is dense in $[0, 1]$.

Let $\lambda \in (0, 1)$ be given. Choose a sequence $(\lambda_n)$ in $(0, 1)$ such that $\lambda_{2n-1} > \lambda_{2n+1} > \lambda > \lambda_{2n+2} > \lambda_{2n}$, $n = 1, 2, \ldots$, and $\lim_n \lambda_{2n-1} = \lambda = \lim_n \lambda_{2n}$. Since $\{\phi(A) : A \subset X\}$ is dense in $[0, 1]$, we can choose a set $A_1 \subset X$ such that $\lambda_1 < \phi(A_1) < \lambda_1$. Similarly, since $\{\phi(A) : A \subset A_1\}$ is dense in $[0, \phi(A_1)]$, there exists $A_2 \subset A_1$ such that $\lambda_2 < \phi(A_2) < \lambda_4$. Again, since $\{\phi(A) : A \subset A_1 \setminus A_2\}$ is dense in $[0, \phi(A_1 \setminus A_2)]$, we can choose $B_3 \subset A_1 \setminus A_2$ such that $\lambda_5 < \phi(A_3 \cup B_3) < \lambda_3$. Set $A_3 = A_2 \cup B_3$. Continue this process. We get a sequence of subsets $(A_n)$ of $X$ such that

$$A_{2n-1} \supset A_{2n+1} \supset A_{2n+2} \supset A_{2n}$$

and

$$\lambda_{2n+1} < \phi(A_{2n-1}) < \lambda_{2n-1}, \quad \lambda_{2n} < \phi(A_{2n}) < \lambda_{2n+2},$$

for $n = 1, 2, \ldots$. Let $A = \cap_{n=1}^\infty A_{2n-1}$. Then $A_{2n-1} \supset A \supset A_{2n}$, and hence, $\lambda_{2n-1} > \phi(A) > \lambda_{2n}$, $n = 1, 2, \ldots$. Thus $\phi(A) = \lim_n \lambda_n = \lambda$ and the proof is completed.

Proof of the Theorem. By Theorem 1 and Lemma 4 of [3], we may assume that $S$ is an infinite amenable group. The Theorem is then a direct consequence of Lemma 2 and Lemma 3.

References


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