ABSTRACT. Let \((X, \| \cdot \|)\) be a normed space, and let \([\cdot, \cdot]\) be any semi-inner-product on it. We show that \((X, \| \cdot \|)\) is strictly convex if and only if \(\|y+z\| > \|y\|\) whenever \([z, y] = 0\) and \(z \neq 0\), and if and only if \([Ax, x] \neq 0\) whenever \(\|I+A\| \leq 1\) and \(Ax \neq 0\). The condition that \([z, y] = 0\) can be replaced by a stronger or weaker condition.

A complex or real normed linear space \((X, \| \cdot \|)\) is strictly convex if each point of the unit sphere is an extreme point of the unit ball. Every normed space has at least one semi-inner-product \([4, \text{Theorem 2, p. 31}]\), i.e., a map \([\cdot, \cdot]\) on \(X \times X\) to \(C\) (resp., to \(R\)) such that

(i) \([x+y, z] = \lambda [x, z] + [y, z]\),

(ii) \([x, x] = \|x\|^2\),

(iii) \([x, y] \leq \|x\| \|y\|\),

for all \(x, y, z \in X\), \(\lambda \in C\) (resp., \(\lambda \in R\)).

For a given semi-inner-product \([\cdot, \cdot]\) on the space, one can say that \(y\) is orthogonal to \(z\) if \([z, y] = 0\); the condition that \(y\) is orthogonal to \(z\) then depends on the choice of semi-inner-product. Nonetheless, we show that if \([\cdot, \cdot]\) is any semi-inner-product on \((X, \| \cdot \|)\), the space is strictly convex if and only if \(\|y+z\| > \|y\|\) whenever \(y\) is orthogonal to \(z \neq 0\), and if and only if \(x\) is never orthogonal to \(Ax \neq 0\) for operators \(A\) such that \(\|I+A\| \leq 1\).

This result still holds if the original orthogonality is replaced by a stronger or weaker form, both of which depend only on the normed space, and the latter of which is equivalent to that of James \([3, \text{p. 265}]\).

The last paragraph contains an application of condition (iv).

Related results have been obtained by Berkson \([1, \text{Theorem 5.1, p. 381, and Lemma 5.3, p. 382}]\), James \([3, \text{Theorem 4.3, p. 275, and Theorem 5.2, p. 279}]\), and Palmer \([5, \text{p. 4}]\).

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Theorem. Let \([\cdot, \cdot]\) be any semi-inner-product on \((X, \|\cdot\|)\). The following conditions are equivalent:

(i) \((X, \|\cdot\|)\) is strictly convex;
(ii) If \(\|y+z\| \leq \|y\|\) and \([z, y] = 0\), then \(z = 0\);
(iii) If \(\|y+z\| = \|y\|\) and \([z, y] = 0\), then \(z = 0\);
(iv) If \(A\) is a bounded linear operator on \(X\), if \(\|I+A\| \leq 1\), and if \([Ax, x] = 0\) for some \(x\) in \(X\), then \(Ax = 0\).

Proof. It will be shown that (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iv) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

(i) \(\Rightarrow\) (ii). Let \((X, \|\cdot\|)\) be strictly convex, \(\|y+z\| \leq \|y\|\), and \([z, y] = 0\). We may assume that \(y \neq 0\). For \(0 \leq t \leq 1\) we have

\[
\|y\|^2 = |[y, y]| = |[y + tz, y]| \leq \|y + tz\| \|y\| \\
\leq (t\|y + z\| + (1 - t)\|y\|)\|y\| \leq \|y\|^2,
\]

whence \(\|y + tz\| = \|y\|\) for all \(t\), \(0 \leq t \leq 1\). Since \((X, \|\cdot\|)\) is strictly convex, it follows that \(z = 0\).

(ii) \(\Rightarrow\) (iv). Trivial.

(iv) \(\Rightarrow\) (iii). Suppose that (iii) does not hold; then there exist \(y \neq 0\), \(z \neq 0\) such that \(\|y+z\| = \|y\|\) and \([z, y] = 0\). Let the bounded operator \(A\) be defined as follows:

\[
Ax = \frac{1}{\|y\|^2} \{[x, y](y + z) - x, \quad x \in X\}.
\]

Then \(\|I+A\| \leq 1\), \(Ay = z \neq 0\) and \([Ay, y] = [z, y] = 0\); thus condition (iv) does not hold.

(iii) \(\Rightarrow\) (i). Suppose that the space is not strictly convex. Then there exist distinct points \(u, v\) and \(w\) in \(X\) such that \(\|u\| = 1\), \(\|v\| \leq 1\), \(\|w\| \leq 1\), and \(u\) is on the line segment between \(v\) and \(w\). Clearly, \(\|v\| = \|w\| = 1\). Let \(y = (v+w)/2\) and \(z = v - y = (v - w)/2\). Since \(\|y\| \leq 1\), \(\|u\| = 1\), and \(u\) is on the segment between \(y\) and \(w\), it follows that \(\|y\| = 1\). Thus

\[
|[z, y] + 1| = |[v - y, y] + [y, y]| = |v, y| \leq 1,
\]

and

\[
|[z, y] - 1| = |[z - y, y]| = |-w, y| \leq 1,
\]

from which it follows that \([z, y] = 0\). Since \(z \neq 0\) and \(\|y+z\| = \|v\| = 1\), (iii) does not hold.

Corollary. The condition that \([z, y] = 0\) can be replaced by the stronger condition that \([z, y] = \{0\}\).
Similarly, \( [z, y] = 0 \) can be replaced by \( 0 \in ([z, y]) \), and \( [Ax, x] = 0 \) by \( 0 \in ([Ax, x]) \).

**Proof.** Some implications are trivial; the others can be seen by a careful reading of the previous proof.

**Remark.** The condition that \( 0 \in ([z, y]) \) is equivalent to a definition of orthogonality due to James [3, p. 265 and Theorem 2.1, p. 268], who noted that in spaces of dimension greater than or equal to three, this orthogonality is symmetric if and only if the norm is given by an inner product [2, Theorem 1, p. 560].

**Application.** Some operators to which condition (iv) is relevant are those which are normal elements of some \( B^* \)-algebra, and whose spectrum lies inside some disc which does not contain 0 as an interior point. For if \( T \) is such an operator and \( D \) is the unit disc, then \( \sigma(T) \) is contained in some disc \( \lambda D - \lambda \), and \( \sigma(I + (1/\lambda)T) \subseteq D \), whence by [6, Lemma 4.8.1 i, p. 240] we have

\[
\|I + (1/\lambda)T\| = \|I + (1/\lambda)T\|_\sigma \leq 1.
\]

Thus if \( [Tx, x] = 0 \) then \( [(1/\lambda)Tx, x] = 0 \), whence \( Tx = 0 \).

**References**


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