DIVISIBLE $H$-SPACES

ROBERT F. BROWN

Abstract. Let $X$ be an $H$-space with multiplication $m$. Define, for $x \in X$, $m_3(x) = m(x, x)$ and $m_k(x) = m(x, m_{k-1}(x))$, for all $k > 2$. If $m_k(x) = y$, then $x$ is called a $k$th root of $y$. The $H$-space $(X, m)$ is divisible if every $y$ in $X$ has a $k$th root for each $k \geq 2$. We prove that if $X$ is a compact connected topological manifold without boundary, then $(X, m)$ is divisible and, in fact, that every $y$ in $X$ has at least $k^\beta$ $k$th roots for each $k \geq 2$, where $\beta$ is the first Betti number of $X$.

A group $G$ is divisible if given any $g \in G$ and any integer $k \geq 2$, there is a solution to $x^k = g$ in $G$. A solution $x$ is called a $k$th root of $g$. In 1940, Hopf proved that a compact connected Lie group $G$ is divisible and, moreover, that for each integer $k \geq 2$, every $g \in G$ has either $k^\lambda$ $k$th roots or an infinite number, where $\lambda$ is the number of generators of the exterior algebra $H^\ast(G)$ (rational coefficients) [4].

By an $H$-space, we mean a triple $(X, m, e)$ where $X$ is a topological space and $m : X \times X \to X$ is a map such that $m(x, e) = m(e, x) = x$ for all $x \in X$. Define $m_k : X \to X$ by setting $m_2(x) = m(x, x)$ and, for each $k > 2$, $m_k(x) = m(x, m_{k-1}(x))$. The $H$-space $(X, m, e)$ is divisible if $m_k$ is onto for all $k \geq 2$. If $x, y \in X$ and $m_k(x) = y$, then $x$ is a $k$th root of $y$. We wish to obtain a result, of the sort Hopf discovered for Lie groups, in the more general setting of $H$-spaces.

Observe first that we cannot expect to prove that a very large class of $H$-spaces is divisible. Let $I$ denote the interval $[0, 1]$ and define $m : I \times I \to I$ by $m(s, t) = |s - t|$, then $(I, m, 0)$ is an $H$-space but $m_2(I) = \{0\}$ so the $H$-space is not divisible. We will prove, however, that an $H$-space $(X, m, e)$ is divisible provided that $X$ is a compact connected manifold without boundary. This result includes many spaces not covered by Hopf’s theorem (see, for example, [1] and [5]).

Even when we can show that $(X, m, e)$ is divisible, there is no hope for $k^\lambda$ as a lower bound on the number of $k$th roots for each $x \in X$, as the following example demonstrates. Let $S^3$ denote the 3-sphere.
and let \( H: S^3 \times S^3 \to S^3 \) be quaternion multiplication. Consider \( S^3 \) as the suspension of the 2-sphere, with vertices \( e = (1, 0, 0, 0) \) and \((-1, 0, 0, 0)\). Let \( \Sigma f: S^3 \to S^3 \) be the suspension of a map \( f: S^2 \to S^2 \) of degree 2 then, by the Hopf Homotopy Theorem, there is a homotopy \( h_t: S^3 \to S^3 \) such that \( h_0 = H_2 \) and \( h_1 = \Sigma f \). Since \( S^3 \) is simply-connected, we may assume that \( h_t \) fixes \( e \). Define \( \Delta(S^3) = \{(x, x) | x \in S^3\} \) and \( A = S^3 \times \{e\} \cup \{e\} \times S^3 \cup \Delta(S^3) \subset S^3 \times S^3 \).

A map \( p: S^3 \times S^3 \times \{0\} \to S^3 \) is defined by

\[
p(x, y, t) = H(x, y) \quad \text{if} \quad t = 0, \\
= h_t(x) \quad \text{if} \quad x = y, \\
= x \quad \text{if} \quad y = e, \\
= y \quad \text{if} \quad x = e,
\]

then \( p \) extends to \( P: S^3 \times S^3 \times I \to S^3 \) by the Homotopy Extension Theorem. Define \( m: S^3 \times S^3 \to S^3 \) by \( m(x, y) = P(x, y, 1) \). Then \((S^3, m, e)\) is an \( H \)-space where \( m_2 = \Sigma f \) so the only square root of \( e \) is \( e \) itself even though, in this case, \( k^3 = 2 \). We will prove, however, that if \((X, m, e)\) is an \( H \)-space where \( X \) is a compact connected manifold without boundary, then each \( x \in X \) has at least \( k^3 \) \( k \)th roots, where \( \beta \) denotes the dimension of \( H^1(X) \).

The author wishes to thank George McCarty and John Miller for useful conversations concerning this paper.

The first thing we shall require is the following computation.

**Lemma.** Let \( A \) be a connected Hopf algebra over \( \Lambda \), a commutative ring with unit, such that \( A \) is isomorphic, as an algebra, to the exterior algebra generated by \( x_1, \cdots, x_\lambda \). Let \( \varphi \) be the product of \( A \) (write \( \varphi(x \otimes y) = xy \)) and \( \psi \) the coproduct. Define \( p_2 = \varphi \psi: A \to A \) and, in general, \( p_k: A \to A \) is defined by \( p_k = \varphi(1 \otimes p_{k-1}) \psi \) for each integer \( k \geq 3 \). Let \( x = x_1 x_2 \cdots x_\lambda \), then, for all \( k \geq 2 \), \( p_k(x) = k^\lambda x \).

**Proof.** Since \( A = \sum A_p \) is a graded \( \Lambda \)-module, for \( x \in A_p \) we define the degree of \( x \) by \( \deg(x) = p \). Order the generators \( x_1, \cdots, x_\lambda \) so that \( \deg(x_i) \leq \deg(x_{i+1}) \) for all \( i = 1, \cdots, \lambda - 1 \). The algebra \( A \) is generated as a \( \Lambda \)-module by all monomials \( y_j = x_{j(1)} \cdots x_{j(r)} \), where \( 1 \leq j(1) < \cdots < j(r) \leq \lambda \). Define the weight \( w(y_j) \) of the monomial \( y_j \) by \( w(y_j) = \deg(x_{j(r)}) \). By definition, \( p_2 = \varphi \psi \) so, for each \( i = 1, \cdots, \lambda \),

\[
p_2(x_i) = \varphi(x_i) = \varphi(x_i \otimes 1 + 1 \otimes x_i + \sum a_j y_j \otimes y_j') = 2x_i + \sum (a_j y_j y_j')
\]
where \(a_j \in \Lambda\), \(w(y_j) < \deg(x_i)\) and \(w(y'_j) < \deg(x_i)\). Since \(\Lambda\) is an exterior algebra, either \(y_j y'_j = 0\), because \(y_j\) and \(y'_j\) have a generator in common, or \(y_j y'_j\) is again a generating monomial (up to sign) and since
\[
w(y_j y'_j) = \max\{w(y_j), w(y'_j)\}
\]
then \(w(y_j y'_j) < \deg(x_i)\). Thus we may write, for \(i = 1, \ldots, \lambda\),
\[
p_i(x_i) = 2x_i + \sum a'_j y'_j
\]
where \(a'_j \in \Lambda\) and \(w(y'_j) < \deg(x_i)\). Suppose that, for \(i = 1, \ldots, \lambda\),
\[
p_{k-1}(x_i) = (k - 1)x_i + \sum a'_j y'_j
\]
where \(a'_j \in \Lambda\) and \(w(y'_j) < \deg(x_i)\), then
\[
p_k(x_i) = \varphi(1 \otimes p_{k-1})y(x_i)
\]
\[
= \varphi(1 \otimes p_{k-1})(x_i \otimes 1 + 1 \otimes x_i + \sum a_j y_j \otimes y'_j)
\]
\[
= x_i + p_{k-1}(x_i) + \sum a_j y_j(p_{k-1}(y'_j))
\]
\[
= kx_i + \sum a'_j y'_j + \sum a_j y_j(p_{k-1}(y'_j)).
\]
Let \(y'_j = x_{j(1)} \cdots x_{j(r)}\), then
\[
p_{k-1}(y'_j) = p_{k-1}(x_{j(1)}) \cdots p_{k-1}(x_{j(r)}).
\]
Of course \(p_{k-1}(x_{j(q)}) \in A_{j(q)}\) so \(p_{k-1}(x_{j(q)})\) is a linear combination of monomials of weight no greater than \(j(q)\), which is less than \(\deg(x_i)\). Therefore \(p_{k-1}(y'_j)\) is a linear combination of monomials of weight less than \(\deg(x_i)\). Since the monomials \(y_j\) are of weight less than \(\deg(x_i)\) and, by the induction hypothesis, the same is true of the \(y'_j\), we have proved, for all integers \(k \geq 2\) and each generator \(x_i\), \(i = 1, \ldots, \lambda\), that
\[
p_k(x_i) = kx_i + \sum a'_j y'_j
\]
where \(a'_j \in \Lambda\) and \(w(y'_j) < \deg(x_i)\). Obviously \(p_k(x_i) = kx_i\) because there are no generators of lower degree. Suppose, for some \(\mu < \lambda\), that
\[
p_k(x_1) \cdots p_k(x_\mu) = k^\mu(x_1 \cdots x_\mu)
\]
then
\[
p_k(x_1) \cdots p_k(x_{\mu+1}) = k^\mu(x_1 \cdots x_\mu) p_k(x_{\mu+1})
\]
\[
= k^{\mu+1}(x_1 \cdots x_{\mu+1}) + k^\mu(x_1 \cdots x_\mu) \sum a_j y_j'('
\]
\[
= k^{\mu+1}(x_1 \cdots x_{\mu+1}) + k^\mu \sum a'_j (x_1 \cdots x_\mu) y'_j.
\]
But \(w(y''_j) < \deg(x_{\mu+1})\) so \(y''_j = x_{j(1)} \cdots x_{j(r)}\) where \(j(q) \leq \mu\) for all \(q = 1, \ldots, r\) because of the order imposed on the \(x_i\). Therefore, since
A is an exterior algebra, \((x_1 \cdots x_\mu) y'_1 = 0\) and we have
\[
p_k(x_1) \cdots p_k(x_{\mu+1}) = k^\mu + 1(x_1 \cdots x_{\mu+1}).
\]
Thus, in a finite number of steps, we obtain
\[
p_k(\bar{x}) = p_k(x_1) \cdots p_k(x_n) = k^\lambda (x_1 \cdots x_\lambda) = k^\lambda \bar{x}.
\]
By an \(H\)-manifold, we shall mean an \(H\)-space \((M, m, e)\) where \(M\) is a compact connected manifold without boundary.

Let \((M, m, e)\) be an \(H\)-manifold and let \(x_1, \cdots, x_\lambda\) generate \(H^*(M)\), then since \(M\) is orientable [3], \(\bar{x} = x_1x_2 \cdots x_\lambda \in H^n(M)\), where \(n\) is the dimension of \(M\). Define \(\Delta : M \to M \times M\) to be the diagonal map. Then \(H^*(M)\) is a connected Hopf algebra over the rationals with product \(\Delta^*\) and coproduct \(m^*\). Observe that we defined \(m_2 = m\Delta\) and the maps \(m_k\) for \(k > 2\) so that the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{m_k} & M \\
\Delta \downarrow & & \downarrow m \\
M \times M & \xrightarrow{1 \times m_{k-1}} & M \times M
\end{array}
\]
commutes. Therefore, by the lemma,
\[
m_k^*(\bar{x}) = p_k(\bar{x}) = k^\lambda \bar{x} \neq 0,
\]
for all \(k \geq 2\).

By [2], since \(m_k^*: H^n(M) \to H^n(M)\) is not the zero homomorphism then, for each \(x \in M\), there is a lower bound for the number of \(k\)th roots of \(x\), namely, the order of the cokernel of the induced homomorphism
\[
m_k^*: \pi_1(M, e) \to \pi_1(M, e).
\]
It is well known that \(m_k^*(\alpha) = k\alpha\) for all \(\alpha \in \pi_1(M, e)\).

By the Fundamental Theorem of Abelian Groups, we write
\[
\pi_1(M, e) \cong Z^{(1)} \oplus \cdots \oplus Z^{(\beta)} \oplus T = F \oplus T
\]
where each \(Z^{(i)}\) is infinite cyclic and \(T\) is finite. By the Hurewicz Isomorphism Theorem and the Universal Coefficient Theorem, \(\beta\) is the dimension of \(H^1(M)\). Let \(m_k^*\) denote the restriction of \(m_k^*\) to \(F\), then the cokernel of \(m_k^*\) is \(Z_k^{(1)} \oplus \cdots \oplus Z_k^{(\beta)}\) where \(Z_k^{(i)}\) is a cyclic group of order \(k\). Therefore, the cokernel of \(m_k^*\) has order \(k^\beta\) and since the order of the cokernel of \(m_k^*\) is at least as large, we have proved
Theorem. Every $H$-manifold $(M, m, e)$ is divisible. Moreover, each $x \in M$ has at least $k^\beta$ $k$th roots for all integers $k \geq 2$, where $\beta$ denotes the dimension of $H^1(M)$.

References