WEAK $A$-CONVEX ALGEBRAS

ALLAN C. COCHRAN

Abstract. Necessary and sufficient conditions are given in terms of $E'$ that a weak topology $w(E, E')$ on an algebra $E$ be $A$-convex. The main condition is that each element $g$ of $E'$ contain a weakly closed subspace $L$ of finite codimension such that $g$ is bounded on all multiplicative translates of $L$. For weak topologies, $A$-convexity (which assumes only separate continuity of multiplication) is equivalent to joint continuity of multiplication.

Let $E$ be an algebra, $E'$ a total subspace of the dual of $E$ and $w(E, E')$ the weak topology of $E$ determined by $E'$. The purpose of this paper is to determine necessary and sufficient conditions that $w(E, E')$ be $A$-convex. Warner [4] has given a necessary and sufficient condition that $w(E, E')$ be locally $m$-convex and also a necessary and sufficient condition that multiplication be jointly ($w(E, E')$) continuous. One of the equivalent forms of our condition is that $w(E, E')$ is $A$-convex (which requires only the separate continuity of multiplication) if and only if multiplication is jointly ($w(E, E')$) continuous. Thus, all weak topological algebras (joint continuity of multiplication) are already $A$-convex. $A$-convex algebras, which include the locally $m$-convex algebras, were introduced in [2]. In §2 the basic properties are given along with some examples. The main results are given in §3.

2. $A$-convex algebras. Throughout this note, $E$ will denote an algebra, $E'$ a total subspace of the dual of $E$ and $w(E, E')$ the weak topology on $E$ induced by $E'$. The proofs of the results given here may be found in [2].

(2.1) Definition. A subset $V$ of $E$ is called $A$-convex if $V$ is absolutely convex, absorbing and for each $x \in E$, $V$ absorbs $xV$ and $Vx$.

The inverse image of an $A$-convex set under a homomorphism is $A$-convex, as is the image of an $A$-convex set under a surjective homomorphism.

(2.2) Definition. An $A$-convex algebra is an algebra $E$ together with a topology on $E$ whose neighborhood system at zero has a basis of $A$-convex sets.

Presented to the Society, January 23, 1970; received by the editors October 3, 1969.

AMS 1968 subject classifications. Primary 4650; Secondary 4601, 4625, 1620.

Key words and phrases. $A$-convex algebra, locally $m$-convex algebra, weak topology, topological algebra.
(2.3) Definition. A seminorm $p$ on $E$ is called $m$-absorbing if for all $x$ in $E$ there are constants $M_x$ and $N_x$ with

(i) $p(xy) \leq M_x p(y)$, for all $y$ in $E$; and

(ii) $p(yx) \leq N_x p(y)$, for all $y$ in $E$.

Thus, an $A$-convex algebra is an algebra $E$ with a topology determined by a family of $m$-absorbing seminorms. $A$-convexity is preserved with respect to taking subspaces, products and quotients. Each $A$-convex algebra can be topologically and algebraically embedded in an $A$-convex algebra with identity. It is clear from the definition that multiplication is separately continuous.

(2.4) Example. Any locally $m$-convex (hence also Banach) algebra is $A$-convex.

(2.5) Example. Let $C[0, 1]$ denote the algebra of continuous real-valued functions on $[0, 1]$ (with pointwise operations). A norm on $C[0, 1]$ is given by

$$p(f) = \sup \{ |f(x)\phi(x)| : x \in [0, 1] \},$$

where $\phi(x) = x$, $0 \leq x \leq \frac{1}{2}$ and $\phi(x) = 1-x$, $\frac{1}{2} < x \leq 1$. $(C[0, 1], p)$ is a normed linear space which is $A$-convex (not locally $m$-convex). An $A$-convex algebra which is normable is called an $A$-normed algebra. The space $(C[0, 1], p)$ is not complete.

(2.6) Example. Let $C_b(\mathbb{R})$ denote the algebra of bounded continuous complex-valued functions on $\mathbb{R}$ (pointwise operations). Let $C_0^+(\mathbb{R})$ denote the set of strictly positive real-valued continuous functions on $\mathbb{R}$ which vanish at infinity. For each $\phi \in C_0^+(\mathbb{R})$, let

$$p_\phi(f) = \sup \{ |f(x)\phi(x)| : x \in \mathbb{R} \}, \quad f \in C_b(\mathbb{R}).$$

Then $p_\phi$ is a seminorm and the topology $\beta$ determined by \( \{ p_\phi : \phi \in C_0^+(\mathbb{R}) \} \) is an $A$-convex topology on $C_b(\mathbb{R})$. This so-called weighted space is a complete $A$-convex algebra with identity which is not locally $m$-convex (see [2], [6]).

(2.7) Theorem. An algebra $E$ with a locally convex linear topology for which multiplication is separately continuous is $A$-convex if and only if it is isomorphic to a subalgebra of a product of $A$-normed algebras.

The relationship between $A$-convex and locally $m$-convex algebras is given by the fact that a barrelled $A$-convex algebra is locally $m$-convex.

For the remainder of this paper the following notations will be used:

For a linear functional $g$ on $E$, $K(g)$ will denote the kernel of $g$. The polar of a set $V$ in $E'$, taken in $E$, will be denoted by $V^\circ$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
3. **Weak \( A \)-convex algebras.** In the proof of the main two theorems the following results will be needed. The proof of Lemma 3.1 may be found in [4] and the proof of Lemma 3.2 is omitted.

(3.1) **Lemma.** Let \( V \) be a \( w(E, E') \)-neighborhood of zero. Then \( L = \cap \{ K(v) : v \in V^0 \} \) is a \( w(E, E') \)-closed subspace of finite codimension.

(3.2) **Lemma.** Let \( g \) be a linear functional on \( E \) and \( L \) a subspace of \( K(g) \). Then \( EL \subseteq K(g) \) is equivalent to the property

(*) for all \( x \) in \( E \) there is a constant \( M_x \) such that \( |g(xL)| \leq M_x \).

Also, \( LE \subseteq K(g) \) is equivalent to

(**) for all \( x \) in \( E \) there is a constant \( N_x \) such that \( |g(Lx)| \leq N_x \).

If \( g \) is a linear functional on \( E \) and \( L \) is a subspace such that (*) and (***) hold, then we say that \( g \) is bounded on the (multiplicative) translates of \( L \).

(3.3) **Theorem.** Let \( E \) be an algebra and \( E' \) a total subspace of the dual of \( E \). Then \( w(E, E') \) is \( A \)-convex if and only if for all \( g \) in \( E' \), \( K(g) \) contains a weakly closed subspace \( L \) of finite codimension such that \( g \) is bounded on all translates of \( L \).

**Proof.** Let \( w(E, E') \) be \( A \)-convex and \( g \) be in \( E' \). Then \( \{g\}^0 \) contains an \( A \)-convex weakly closed neighborhood \( V \) of zero. Let \( L = \cap \{ K(v) : v \in V^0 \} \). By Lemma 3.1, \( L \) is a weakly closed subspace of finite codimension. Clearly \( g \) is in \( V^0 \), \( L \subseteq V^{00} \) and \( V = V^{00} \) so \( L \subseteq K(g) \).

For \( x \) in \( E \), the \( A \)-convexity of \( V \) insures the existence of constants \( M_x \) and \( N_x \) such that \( |g(xL)| \leq M_x \) and \( |g(Lx)| \leq N_x \). Hence \( g \) is bounded on all translates of \( L \) and the condition is necessary.

Let \( g (\neq 0) \) be in \( E' \) and \( L \) a weakly closed subspace of finite codimension with \( L \subseteq K(g) \) and \( g \) bounded on all translates of \( L \). It suffices to show that \( \{g\}^0 \) contains an \( A \)-convex weak neighborhood of zero. By the induced map theorem there exists a unique continuous linear functional \( \bar{g} \) on \( F = E/L \) such that \( g = \bar{g} \circ \phi_1 \), where \( \phi_1 \) denotes the quotient map from \( E \) to \( F \). Since \( F \) is a finite dimensional Hausdorff space, it is normable. We may assume that the norm on \( F \) is chosen so that \( \bar{g}(\hat{V}) = \{z : |z| \leq 1\} \) where \( \hat{V} \) is the unit ball in \( F \). The map \( \phi_1 \) is a continuous linear functional so \( V = \phi_1^{-1}(\hat{V}) \) is an absolutely convex absorbing neighborhood of zero. Then

\[
|g(V)| = |\bar{g}(\phi_1(V))| = |\bar{g}(\hat{V})| \leq 1, \quad \text{so } V \subseteq \{g\}^0.
\]

We now show that \( V \) is \( A \)-convex: For \( x \) in \( E \), \( xL \subseteq K(g) \) by Lemma
3.2 and \([xL]^-\) is a closed subspace contained in \(K(g)\). By the induced map theorem there is a unique continuous linear functional \(g^*\) on \(H = E/\,[xL]^-\) with \(g = g^* \circ \phi_2\) where \(\phi_2\) is the quotient map of \(E\) to \(H\). The map \(t_x\) of \(F\) to \(H\) defined by \(t_x(\mathcal{y}) = [xy]^\sim\) is a well-defined linear map from the finite dimensional space \(F\) to \(H\). Thus, \(t_x\) is continuous and the image of \(F\) is a finite dimensional Hausdorff space. The image of \(F\) under \(t_x\) is normable and the norm may be chosen so that \(|g^*(V^*)| \leq 1\), where \(V^*\) denotes the unit ball in \(H\). It follows from a standard theorem that there exists a constant \(M_x\) such that

\[
\|t_x(\mathcal{y})\|_2 \leq M_x\|\mathcal{y}\|, \quad \text{for all } \mathcal{y} \text{ in } F,
\]

where \(\| \|\) denotes the norm on \(t_x(F)\) and \(\| \|\) denotes the norm on \(F\). If \(y\) is in \(V\), \(\mathcal{y}\) is in \(\hat{V}\) and \(\| [xy]^\sim \|_2 \leq M_x\). Thus,

\[
|g^*(M_x^{-1}[xy]^\sim)| \leq 1 \quad \text{so} \quad |g(M_x^{-1}xy)| \leq 1.
\]

This shows that \(xV \subseteq M_xV\). Similarly, there is a constant \(N_x\) with \(Vx \subseteq N_xV\). Hence \(V\) is \(A\)-convex and the condition is sufficient.

The result of Theorem 3.3 gives another interesting consequence: \(A\)-convexity is equivalent to joint \((w(E, E'))\)-continuity of multiplication. Hence any weak topological algebra (joint continuity) is an \(A\)-convex algebra.

**Theorem 3.4.** Let \(E\) be an algebra. Then \(w(E, E')\) is \(A\)-convex if and only if the multiplication of \(E\) is jointly \((w(E, E'))\) continuous.

**Proof.** Theorem 3.3 combined with Lemma 3.2 give the result that \(w(E, E')\) is \(A\)-convex if and only if for all \(g\) in \(E'\), \(K(g)\) contains a weakly closed subspace \(L\) of finite codimension with \(EL \subseteq K(g)\) and \(LE \subseteq K(g)\). From a theorem of Warner [4, Theorem 2] this is equivalent to the joint continuity of multiplication.

**Corollary 3.5.** Let \(E\) be a topological algebra with respect to a weak topology. Then \(E\) is \(A\)-convex.

The following problem remains unsolved:

**Problem 3.6.** Is there an example of an algebra \(E\) and a subspace \(E'\) of its dual such that \(w(E, E')\) is \(A\)-convex but not locally \(m\)-convex?

**References**


University of Arkansas, Fayetteville, Arkansas 72701