A NOTE ON CONNECTED AND PERIPHERALLY CONTINUOUS FUNCTIONS

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Abstract. In this paper it is proved that under certain conditions on the domain and range spaces an open monotone connected function preserves unicoherentness and hereditary local connectedness. In addition, a monotone-light factorization theorem is proved for certain connected functions and peripherally continuous functions.

1. Introduction. Many properties of continuous functions are also possessed by certain noncontinuous functions (see for example [1]–[4] and [6]). This paper is concerned with three of these properties: namely, preservation by an open monotone connected function of unicoherentness and hereditary local connectedness, and factorization of connected and peripherally continuous functions.

Some definitions will now be recalled. Let X and Y be topological spaces. A function \( f : X \to Y \) is connected if whenever \( A \) is a connected set in \( X \), then \( f(A) \) is connected in \( Y \) [6, p. 488]. The function \( f \) is peripherally continuous if whenever \( p \) is a point in \( X \) and \( U \) and \( V \) are open sets containing \( p \) and \( f(p) \), respectively, there exists an open set \( W \) such that \( x \in W \subseteq U \) and \( f \) maps the boundary of \( W \) into \( V \) [2, p. 751]. The function \( f \) is monotone if for each \( y \in Y \), \( f^{-1}(y) \) is connected.

If \( A \) is a set, \( \text{cl}(A) \) will denote the closure of \( A \) and \( F(A) \) will denote the boundary of \( A \). The notation \( x_n \to x \) is used for a sequence \( \{x_n\} \) converging to \( x \). If \( A \) is a collection of sets, \( A^* \) will denote the point set union of the sets in \( A \).

2. Connected functions. In Theorems 1 and 2 the only assumptions concerning the function \( f \) are that \( f \) is open in Theorem 1 and \( f \) is open and monotone in Theorem 2. These theorems are then used in Theorems 3 and 4 to obtain the results that, under certain conditions, unicoherentness and hereditary local connectedness are preserved by an open monotone connected function.

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Theorem 1. If \( f \) is an open function from the topological space \( X \) onto the topological space \( Y \) and \( y_n \to y \) in \( Y \), then \( f^{-1}(y) \subseteq \lim \inf f^{-1}(y_n) \).

Proof. If \( x \in f^{-1}(y) \) and \( U \) is an open set in \( X \) such that \( x \in U \) and \( U \) fails to intersect \( f^{-1}(y_n) \) for infinitely many \( n \), then since \( f \) is open \( f(U) \) is an open set containing \( y \) and \( f(U) \) fails to contain \( y_n \) for infinitely many \( n \). This contradicts \( y_n \to y \).

Theorem 2. If \( f \) is an open monotone function from the topological space \( X \) onto the first countable space \( Y \) and \( E \) is connected in \( Y \), then \( f^{-1}(E) \) is connected in \( X \).

Proof. Suppose \( f^{-1}(E) = H \cup K \), where \( H \) and \( K \) are separated. Then \( E = f(H) \cup f(K) \) and \( f(H) \) and \( f(K) \) are not separated. Suppose \( q \) is a point in \( f(H) \cap f(K) \). Then for some \( r \) in \( H \) and \( t \) in \( K \), \( f(r) = f(t) = q \). Hence \( r \) and \( t \) are in \( f^{-1}(q) \). Since \( f \) is monotone \( f^{-1}(q) \) is connected. Therefore either \( f^{-1}(q) \subseteq H \) or \( f^{-1}(q) \subseteq K \) since \( H \) and \( K \) are separated. But \( r \in f^{-1}(q) \subseteq H \) and \( t \in f^{-1}(q) \subseteq K \). This contradiction shows that \( f(H) \cap f(K) = \emptyset \). Thus, one of \( f(H) \) and \( f(K) \) must contain a limit point of the other. Suppose \( q \) is in \( f(H) \) and is a limit point of \( f(K) \). Since \( Y \) is first countable there is a sequence \( \{q_n\} \) in \( f(K) \) such that \( q_n \to q \). By Theorem 1, \( f^{-1}(q) \subseteq \lim \inf f^{-1}(q_n) \). Hence, if \( x \) is in \( f^{-1}(q) \), then \( x \) is a limit point of \( K \) since \( f^{-1}(q_n) \subseteq K \) for all \( n \). But \( x \in H \) since \( f^{-1}(q) \subseteq H \). Thus, \( H \) and \( K \) are not separated. This contradiction shows that \( f^{-1}(E) \) is connected.

Theorem 3. If \( f \) is an open monotone connected function from the unicoherent \( T_2 \) continuum \( X \) onto the first countable compact \( T_2 \) space \( Y \), then \( Y \) is a unicoherent continuum.

Proof. Since \( f \) is a connected function, then \( Y \) is connected and hence a continuum by the virtue of being compact. Suppose \( Y = H \cup K \), where \( H \) and \( K \) are continua. Then \( X = f^{-1}(H) \cup f^{-1}(K) \). By Theorem 2, \( f^{-1}(H) \) and \( f^{-1}(K) \) are connected. By Theorem 2.1 of [4], \( f^{-1}(H) \) and \( f^{-1}(K) \) are closed and hence are continua since \( X \) is compact. Since \( X \) is unicoherent \( f^{-1}(H) \cap f^{-1}(K) \) is a continuum. Now \( f^{-1}(H) \cap f^{-1}(K) = f^{-1}(H \cap K) \) and since \( f \) is connected, \( f(f^{-1}(H \cap K)) = H \cap K \) is connected. Thus, \( H \cap K \) is a closed connected subset of the compact space \( Y \) and is therefore a continuum. This shows that \( Y \) is unicoherent.

Theorem 4. If \( f \) is an open monotone connected function from \( X \) onto \( Y \), where \( X \) and \( Y \) are separable metric continua and \( X \) is hereditarily locally connected, then \( Y \) is hereditarily locally connected.
Proof. By Theorem 2.1 of [7, p. 89] a continuum is hereditarily locally connected if and only if it has no nondegenerate continuum of convergence. Therefore, suppose K is a nondegenerate continuum of convergence in Y. Let \{K_n\} be a sequence of disjoint continua none intersecting K such that lim K_n = K, and let p and q be distinct points in K. By Theorem 7 of [7, p. 11] some subsequence \{f^{-1}(K_{n_i})\} converges to a limiting set L. Since X is compact, L \neq \emptyset. Hence, by Theorem 9.1 of [7, p. 14], L is a continuum. Let x \in f^{-1}(p) and U an open set in X containing x. Since f is an open function, f(U) is open and contains p. Therefore f(U) intersects K_n for all but a finite number of n. Hence, U intersects f^{-1}(K_{n_i}) for all but a finite number of i. Thus, x \in L and hence f^{-1}(p) \subseteq L. Similarly, f^{-1}(q) \subseteq L. Since neither f^{-1}(p) and f^{-1}(q) is empty and they are disjoint, it follows that L is nondegenerate. Therefore X contains a nondegenerate continuum of convergence. This contradicts X being hereditarily locally connected. Thus, Y is hereditarily locally connected.

3. Factorization. If f is a function from the space X onto the space Y, let X' denote the collection of all components of sets f^{-1}(y), where y varies over Y. The collection X' will be called the component decomposition of X induced by f. Theorems 5 and 6 give a factorization of f analogous to that given in Theorems 3.6 and 3.7 of [1].

Theorem 5. If f is a connected function from the compact, separable metric space X onto the regular T_2 space Y, and the component decomposition X' of X induced by f is upper semicontinuous, then f can be factored into the composite f = f_2 f_1, where f_1 from X onto X' is monotone and continuous and f_2 from X' onto Y is light and connected.

Proof. Define f_1(x) = C if and only if x is a point in C, where C is a component of some f^{-1}(y), y in Y. By [7, p. 127], f_1 is monotone and continuous. Define f_2(C) = y if and only if C is a component of f^{-1}(y). Then f_2 is light since the elements of f_2^{-1}(y) are the components of f^{-1}(y) and these form a totally disconnected set in X'. For if H is any nondegenerate subcollection of f_2^{-1}(y), then H is connected in X' if and only if H* is connected in X [5, p. 275]. Now H being nondegenerate implies H* contains more than one component of f^{-1}(y) and hence is not connected. Thus H is not connected and f_2 is therefore light.

By definition of f_1 and f_2, f = f_2 f_1. Therefore it remains to show that f_2 is connected.

To this end let A be a connected subset of X'. Then A* is connected in X. Since f is a connected function f(A*) is connected. But f(A*)
=f_2f_1(A) = f_2(A)$. Hence, \( f_2(A) \) is connected and \( f_2 \) is a connected function.

**Theorem 6.** If \( f \) is a peripherally continuous function from the compact separable metric space \( X \) onto the regular \( T_2 \) space \( Y \) and the component decomposition \( X' \) of \( X \) is upper semicontinuous, then \( f \) can be factored into the composite \( f = f_2f_1 \), where \( f_1 \) and \( f_2 \) are defined as in Theorem 5, \( f_1 \) is monotone and continuous, and \( f_2 \) is light and peripherally continuous.

**Proof.** That \( f_1 \) is monotone and continuous and \( f_2 \) is light follows as in Theorem 5.

Since \( f \) is peripherally continuous, the components of \( f^{-1}(y), y \) in \( Y \), are closed by Theorem 1 of [3, p. 639]. Hence the elements of \( X' \) are continua since \( X \) is compact.

Let \( g \) be an element in \( X' \) and \( f(g) = y \), and let \( U \) and \( V \) be open in \( X' \) and \( Y \) containing \( g \) and \( y \), respectively. Then \( U^* \) is open in \( X \) and \( g \subseteq U^* \). Since \( f \) is peripherally continuous, for each \( x \) in \( g \) there is an open set \( w_x \) in \( X \) such that \( x \in W_x \subseteq U^* \) and \( f(F(W_x)) \subseteq V \). The collection \( \{W_x: x \in g\} \) is an open covering of \( g \). Since \( g \) is compact some finite subcollection \( \{W_1, \ldots, W_n\} \) covers \( g \). Let \( W = \bigcup_{i=1}^n W_i \). Then \( g \subseteq W \subseteq U \) and \( f(F(W)) \subseteq V \).

Let \( H \subseteq X' \) such that \( k \) is in \( H \) if and only if \( k \subseteq W \). Then \( H \) is open in \( X' \) and \( H \subseteq U \). Let \( h \in F(H) \) and suppose \( h \cap F(W) = \emptyset \). Now \( h \subseteq \text{cl}(W) = W \cup F(W) \) and \( W \cap F(W) = \emptyset \). Hence, \( h \subseteq W \) and thus \( h \in H \). This contradicts \( h \in F(H) \). Therefore, \( h \cap F(W) \neq \emptyset \). Let \( x \in h \cap F(W) \). Then \( f(x) = f(h) = f_2(h) \) and since \( x \in F(W) \), \( f(x) \in V \). Thus, \( f_2(h) \in V \) and \( f_2(F(H)) \subseteq V \). This shows that \( f_2 \) is peripherally continuous.

**Remark.** In [1] a proof is given that under rather restrictive conditions on the space \( X \), the component decomposition \( X' \) of \( X \) induced by a peripherally continuous function is indeed upper semicontinuous. If this could be proven under milder restrictions for connected and peripherally continuous functions, then Theorems 5 and 6 would more closely resemble the monotone-light factorization theorem for continuous functions [7, p. 143].

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**References**


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