AN UPPER ASYMPTOTIC ESTIMATE OF BROWNIAN PATH VARIATION

OLAF P. STACKELBERG

Abstract. Let \( X(t, \omega) \) be standard Brownian motion, and denote by \( \mathfrak{P}_n \) the family of all partitions of \([0, 1]\) with minimum distance between successive partition points \( \geq 1/n \). Then

\[
P \left[ \limsup_{n \to \infty} \sup_{P \in \mathfrak{P}_n} \frac{\log \log n}{\log^2 n} \sum_{i \in P} (X(t_i) - X(t_{i-1}))^2 \leq K \right] = 1
\]

where \( K \) is an appropriate constant.

Let \( \{X(t, \omega); 0 \leq t \leq 1\} \) be standard Brownian motion. As is customary, we will suppress the \( \omega \) and write \( X(t) \) for \( X(t, \omega) \) and \( \{\cdots\} \) for \( \{\omega|\cdots\} \). Paul Lévy has shown \([3]\) that for a sequence of refining partitions \( \{P_n\} \) of \([0, 1]\) which becomes dense in \([0, 1]\)

\[
P \left[ \lim_{n \to \infty} \text{var}(X, P_n) = 1 \right] = 1,
\]

where \( P = (t_0, t_1, \cdots, t_m); \ 0 \leq t_0 < t_1 < \cdots < t_m \leq 1; \ \text{var}(X, P) = \sum_{i=1}^{m} (X(t_i) - X(t_{i-1}))^2 \). For a martingale proof of this fact, see J. L. Doob \([1, \text{pp. 395ff.}]\).

On the other hand, if the refining and density conditions on the partitions are dropped, then \([3]\)

\[
P \left[ \sup_{P \in \mathfrak{P}} \text{var}(X, P) = \infty \right] = 1,
\]

where \( \mathfrak{P} \) is the family of all finite partitions of \([0, 1]\).

Let \( \mathfrak{P}_n \) be the family of all partitions of \([0, 1]\) with minimum distance between successive partition points \( \geq 1/n \). Recently, P. E. Greenwood has shown \([2]\) that

\[
P \left[ \limsup_{n \to \infty} \sup_{P \in \mathfrak{P}_n} \frac{1}{n^2} \text{var}(X, P) \leq K \right] = 1
\]

for an appropriate constant \( K \). It is the purpose of this note to sharpen this result by proving the following

\[\text{Received by the editors December 19, 1969.}\]

\[\text{AMS 1969 subject classifications. Primary 6062.}\]

\[\text{Key words and phrases. Brownian motion, sample path variation, asymptotic estimate.}\]

\[\text{1 The author is a visiting staff member at the University of Illinois.}\]
ASYMPTOTIC ESTIMATE OF BROWNIAN PATH VARIATION

Theorem.

\[
P \left( \lim_{n \to \infty} \sup_{P \in \mathcal{F}_n} \sup \frac{\log \log n}{\log^2 n} \text{var}(X, P) \leq K \right) = 1
\]

for an appropriate constant \( K \).

Remarks. The method of proof of our theorem parallels that of [2], with the improvement following from the sorting of the intervals \( \Delta t_i \) according to their size. An immediate improvement of the result in [2] can be realized by using the Schwarz inequality to replace the \( n^2 \) by \( n \) in Lemma 4, allowing \( 1/n \) to replace \( 1/n^2 \) in the theorem of [2]. It would be interesting to learn how good our estimate is, perhaps by developing a good lower asymptotic estimate. This seems to be a difficult problem.

Proof. Suppose \( P \in \mathcal{F}_n \) with partition points \( 0 = t_0 < t_1 < \cdots < t_r = 1 \), and set \( \Delta t_i = t_i - t_{i-1} \). For \( n > e^* \), define classes \( C_1, \cdots, C_{\psi(n)} \) as follows:

\[
C_1 = \left \{ \Delta t_i: \frac{1}{2[\log n]} < \Delta t_i \leq 1 \right \},
\]

\[
C_2 = \left \{ \Delta t_i: \frac{1}{2[\log^2 n]} < \Delta t_i \leq \frac{1}{2[\log n]} \right \},
\]

\[
C_k = \left \{ \Delta t_i: \frac{1}{2[\log^k n]} < \Delta t_i \leq \frac{1}{2[\log^{k-1} n]} \right \},
\]

\[
C_{\psi(n)} = \left \{ \Delta t_i: \frac{1}{2n} \leq \Delta t_i \leq \frac{1}{2[\log^{\psi(n)-1} n]} \right \},
\]

where \( \psi(n) = \lfloor \log/(\log \log n) \rfloor + 1 \). Then

\[
\frac{1}{[\log^{\psi(n)} n]} < \frac{1}{n} \leq \frac{1}{[\log^{\psi(n)-1} n]}.
\]

The 2 in the denominators above are for convenience of estimations later on.

Now, setting

\[
\varphi(n) = \log^* n / (\log \log n),
\]
\[ P[\text{var}(P, X) > K\varphi(n)] \]

\[ = P \left[ \sum_{k=1}^{\psi(n)} \sum_{c_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n) \right] \]

\[ \leq \sum_{k=1}^{\psi(n)} P \left[ \sum_{c_k} (X(t_i) - X(t_{i-1}))^2 > K\varphi(n)\psi^{-1}(n) \right]. \]

The symbol \( \sum_{c_k} \) means the sum over all those indices \( i \) with \( \Delta t_i \in C_k \).

Consider the inside sum \( \sum_{c_k} \) of (1). Break \([0, 1]\) into equal subintervals of the form

\[ \left[ \frac{(l-1)}{2\log^k n}, \frac{l}{2\log^k n} \right] \]

for \( l = 1, 2, \ldots, 2\log^k n \).

No more than \( \log n + 1 \) points of the form \( p^{(k)}_i \) lie in any \( \Delta t_i \) of \( C_k \) for \( k = 2, 3, \ldots, \psi(n) \), and less than \( 2 \log n \) points of the form \( p^{(1)}_i \) lie in any \( \Delta t_i \) of \( C_1 \). Fix \( k \) and suppose for some \( \Delta t_i \) in \( C_k \) that \( \Delta t_{i-1} \leq p^{(k)}_i \leq \cdots \leq p^{(k)}_{i+\log n} \leq t_i \). Then, with \( n > e^e \), so that \( 2\log n > \log n + 1 \),

\( (X(t_i) - X(t_{i-1}))^2 \]

\[ \leq 2 \log n \left[ (X(t_i) - X(p^{(k)}_{i+m_i}))^2 + (X(p^{(k)}_{i+m_i}) - X(p^{(k)}_{i+m_i-1}))^2 \right. \]

\[ + \cdots + (X(p^{(k)}_{t_i}) - X(t_{i-1}))^2 \]

since by the Schwarz inequality, \( (a_r - a_1)^2 \leq r \sum_{j=1}^{r-1} (a_j - a_{j-1})^2 \) for any \( r \) real numbers \( a_1, a_2, \ldots, a_r \). We sum the left-hand side of (2) over all \( \Delta t_i \) in \( C_k \). On the right-hand side we replace the existing \( t_i \)'s by a point \( t \) in the same interval \([p^{(k)}_{t_i}, p^{(k)}_{t}]\) so as to maximize \( (X(p^{(k)}_t) - X(t))^2 + (X(t) - X(p^{(k)}_{t-1}))^2 \); and we insert such a point \( t \) into all those intervals with no \( t_i \) in them. Then we sum the right-hand side over all the points \( p^{(k)}_i \) and \( t \), and we get

\[ \sum_{c_k} (X(t_i) - X(t_{i-1}))^2 \]

\[ \leq 2 \log n \sum_{t=1}^{2\log n} \left( (X(p^{(k)}_t) - X(t))^2 + (X(t) - X(p^{(k)}_{t-1}))^2 \right). \]

The last expression is independent of the partition \( P \) we started with. The class \( C_k \) could have been empty to start with, but the inequality holds nevertheless.

Now we have, from (1) and (3),
\[ P \left[ \sup_{P \in \mathcal{P}_n} \text{var}(X, P) > K \varphi(n) \right] \]
\[ = P \left[ \sup_{P \in \mathcal{P}_n} \sum_{k=1}^{\psi(n)} \sum_{c_k} (X(t_k) - X(t_{k-1}))^2 > K \varphi(n) \right] \]
\[ \leq \sum_{k=1}^{\psi(n)} P \left[ \sup_{P \in \mathcal{P}_n} \sum_{c_k} (X(t_k) - X(t_{k-1}))^2 > K \varphi(n) \psi^{-1}(n) \right] \]
\[ \leq \sum_{k=1}^{\psi(n)} P \left[ 2 \log n \sum_{l=1}^{2[\log^k n]} \sup_{t \in [b, c]} \left[ (X(p^{(l)}_i) - X(t))^2 + (X(t) - X(p^{(l)}_{i-1}))^2 \right] > K_1 \log n \right] \]

where \( K_1 \) is a constant with \( K \varphi(n) \psi^{-1}(n) > K_1 \log n \). Since \( n > e^t \) we can take \( K_1 = K/2 \).

For integers \( k, 1 \leq k \leq \psi(n) \), set

\[ Y_{k, l} = \sup_{t \in [b, c]} \left[ (X(p^{(l)}_i) - X(t))^2 + (X(t) - X(p^{(l)}_{i-1}))^2 \right] \]

For each \( k \), the \( Y_{k, l}, l = 1, 2, \ldots, 2[\log^k n] \), are independent, identically distributed random variables. By standard Brownian motion arguments (see Lemmas 1 and 2 of [2]) one can show that

\[ P[Y_{k, l} > x] \leq 2P[8(X(p^{(l)}_i) - X(p^{(l)}_{i-1}))^2 > x] \]

and \((X(p^{(l)_i}) - X(p^{(l)}_{i-1}))^2, l = 1, 2, \ldots, 2[\log^k n], \) are independent, identically distributed variables. Putting (6) another way, if the \( Y_{k, l} \) have \( F_k \) as distribution function, and the common distribution of the variables \( 8(X(p^{(l)}_i) - X(p^{(l)}_{i-1}))^2 \) is \( G_k \), then \( 1 - F_k(x) \leq 2(1 - G_k(x)) \) for each \( k \) and all \( x \). By induction (see Lemma 3 in [2]), for a positive integer \( m \),

\[ 1 - F_k^m(x) \leq 2^m(1 - G_k^m(x)). \]

The variables \( [\log^k n]^{1/2}(X(p^{(l)}_i) - X(p^{(l)}_{i-1})) \), \( l = 1, 2, \ldots, 2[\log^k n], \) are independent, normally distributed each with mean 0 and variance \( \frac{1}{2} \). Hence the sum of their squares has a \( \chi^2 \)-type distribution and we get from (6) and (7) that
\[ P \left[ \sum_{l=1}^{2[\log^k n]} Y_{k,l} > x \right] \]

(8)

\[ \leq 2^{2[\log^k n]} P \left[ 8 \sum_{l=1}^{2[\log^k n]} \left( X(p_l) - X(p_{l-1}) \right)^2 > x \right] \]

\[ = \frac{2^{2[\log^k n]}}{\Gamma([\log^k n])} \int_{(x/8)[\log^k n]}^{\infty} e^{[\log^k n]-1} e^{-t} dt. \]

Then, by (4), (5) and (8),

\[ P \left[ \sup_{p \in \mathcal{P}_n} \text{var}(X, P) > K\varphi(n) \right] \]

(9)

\[ \leq \sum_{k=1}^{\psi(n)} \sum_{l=1}^{2[\log^k n]} P \left[ \sum_{l=1}^{2[\log^k n]} Y_{k,l} > \frac{K_1}{2} \right] \]

\[ \leq \sum_{k=1}^{\psi(n)} \frac{2^{2[\log^k n]}}{\Gamma([\log^k n])} \int_{K_2[\log^k n]}^{\infty} e^{[\log^k n]-1} e^{-t} dt \]

where \(32K_2 = K\). If we could show that the last expression in (9) is the \(n\)th term of a convergent series, then by the convergence part of the Borel-Cantelli Lemma, the event \(\sup_{p \in \mathcal{P}_n} \text{var}(X, P) > K\varphi(n)\) would occur only finitely often with probability one, and our theorem would be proven.

We estimate the integral in (9) by repeatedly integrating by parts. For fixed \(k\), and setting \(m = [\log^k n]\) for notational convenience, we get

\[ \int_{K_2 m}^{\infty} t^{m-1} e^{-t} dt = e^{-K_2 m (K_2 m)^{m-1}} \left( 1 + \frac{m - 1}{K_2 m} + \cdots + \frac{(m - 1)!}{(K_2 m)^{m-1}} \right) \]

\[ \leq e^{-K_2 m (K_2 m)^{m-1}} \frac{m!}{m^m} \sum_{l=0}^{m} \frac{m^l}{l!} \quad \text{for } K_2 \geq 1. \]

Now \(\sum_{l=0}^{m} (m!/l!) \sim \frac{1}{2} e^m\), and it follows that, for \(K_2 \geq 1\),

\[ \frac{2^{2m}}{\Gamma(m)} \int_{K_2 m}^{\infty} t^{m-1} e^{-t} dt \leq \frac{2^{2m}}{m!} e^{-K_2 m (K_2 m)^{m}} \frac{m!}{m^m} \sum_{l=0}^{m} \frac{m^l}{l!} \]

(10)

\[ \leq \exp \left\{ (2 \log 2 - K_2 + 1 + \log K_2) m \right\} \]

This relation can be seen by using the Central Limit Theorem for a sequence of independent, identically distributed Poisson random variables with parameter 1.

Also, the relation follows from the theory of Szász operators [4].
for large $m$, i.e. for large $n$. We take $K_2$ large enough so that $2 \log 2 - K_2 + 1 + \log K_2 < -1$, and $n_0 > e^e$ and large enough so that the estimates (10) hold for $n > n_0$. Then

$$
\sum_{n=n_0}^{\infty} \sum_{k=1}^{\psi(n)} \frac{2^{[\log^k n]}}{\Gamma([\log^k n])} \int_{K_2[\log^k n]}^{\infty} t^{[\log^k n]-1} e^{-t} dt
$$

$$
\leq \sum_{n=n_0}^{\infty} \psi(n) \exp\{(2 \log 2 - K_2 + 1 + \log K_2)[\log n]\} < \infty,
$$

and this proves our theorem.

REFERENCES


