FINITE OPERATORS

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ABSTRACT. A bounded linear operator A on a Hilbert space H is called finite if $\|AX-XA-1\|\geqq 1$ for each $X\in B(H)$. The class of finite operators is uniformly closed, contains every normal operator, every operator with a compact direct summand, and the entire C^*-algebra generated by each of its members. These results imply that the set of operators with a finite dimensional reducing subspace is not uniformly dense. It is also shown that the set of self-commutators is uniformly closed.

Introduction. If A is a linear operator on a finite dimensional complex Hilbert space H, then every commutator of the form $AX-XA$ has trace 0 and consequently 0 belongs to the numerical range $W(AX-XA)$. However, if H is infinite dimensional, then, as shown by Halmos [2], there exist bounded operators A and B on H such that $W(AB-BA)$ is a vertical line segment in the open right half-plane. The present paper initiates a study of the class $\mathfrak{F}$ of operators A on H which have the property that $0\in W(AX-XA)^-$ for every bounded operator X on H. We call such operators finite, the term being suggested by the facts that $\mathfrak{F}$ contains all normal operators, all compact operators, all operators having a direct summand of finite rank, and the entire C^*-algebra generated by each of its members.

The information obtained about finite operators (§3) is meager, but seems to include the previous work on the subject. These results suggest more questions than they answer; some of these are listed in §4.

The main technique is based on an orthogonality interpretation of the numerical range of an element of a Banach algebra introduced in [8]. This point of view is developed in §1 and is used to give more natural proofs of theorems in [8] and [6]. It is also used to prove that the set of self-commutators on H is norm closed.

1. **Orthogonality and the numerical range.** Let $\mathfrak{B}$ be a complex Banach algebra with identity and let $\mathcal{P}=\{f\in\mathfrak{B}^*:f(1)=1=\|f\|\}$ be the set of normalized positive functionals (states) on $\mathfrak{B}$. If $A\in\mathfrak{B}$, then the numerical range [8] of A is by definition the set $W_0(A)$

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$=\{f(A):f\in\mathcal{P}\}$. $W_0(A)$ is convex, compact and contains the spectrum of A. Also, if $\mathcal{B}=\mathcal{B}(H)$ is the algebra of bounded operators on a complex Hilbert space H, then $W_0(A)=W(A)^-$ is precisely the closure of the ordinary numerical range of A, i.e., the numbers (Ax, x) with $\|x\|=1$.

THEOREM 1. *Let $A\in\mathcal{B}$. Then $0\in W_0(A)$ if and only if $|\lambda|\leqq\|A-\lambda\|$ for all complex numbers λ.*

PROOF. This result appears in [8], but it admits a much simpler proof: If $f(A)=0$ for some $f\in\mathcal{P}$, then $\lambda=f(\lambda-A)$ has modulus at most $\|f\|\|\lambda-A\|=\|\lambda-A\|$ for any complex λ. Conversely, if the condition of the theorem is satisfied, then one can define a linear functional f on the span of 1 and A by $f(\alpha A+\beta)=\beta$. Clearly f has norm at most 1 and hence has an extension f to $\mathcal{B}$ with norm $\leqq 1$. Since $f(1)=1$, the extension belongs to $\mathcal{P}$. Finally, $f(A)=0$ by definition of f, so that $0\in W_0(A)$.

COROLLARY 1. $W_0(A)=\bigcap_{\mu\in C}\{\lambda:|\lambda-\mu|\leqq\|A-\mu\|\}$.

COROLLARY 2. *$W_0(A)=\mathcal{C}(\sigma(A))=$ convex hull of the spectrum of A if and only if $\|A-\lambda\|=|\sigma(A-\lambda)|=$ spectral radius of $A-\lambda$ for all complex λ.*

PROOF. Both hypothesis and conclusion are equivalent to the assertion that every closed disk that contains $\sigma(A)$ also contains $W_0(A)$.

If x and y are vectors in a complex inner-product space, then $x\perp y$ if and only if $\|\lambda y\|\leqq\|x-\lambda y\|$ for all complex numbers λ. This latter condition makes sense in any normed space and therefore may be taken as the definition of the relation $x\perp y$. With this convention, Theorem 1 asserts that $0\in W_0(A)$ if and only if $A\perp 1$. More generally, Corollary 1 shows that $W_0(A)$ consists of those complex numbers λ for which $(A-\lambda)\perp 1$.

The orthogonality relation just introduced is not symmetric in general, and so it is natural to ask what $1\perp A$ means. This is the content of the following theorem.

THEOREM 2. *Let $\mathcal{B}$ be a C^*-algebra with identity. These are equivalent conditions on an element A of $\mathcal{B}$:*

(i) $\|A\|\leqq\|A-\lambda\|$ *for all complex* λ.

(ii) *There exists $f\in\mathcal{P}$ such that $f(A^*A)=\|A\|^2$ and $f(A)=0$.*

(iii) $\|A\|^2+|\lambda|^2\leqq\|A+\lambda\|^2$ *for all complex* λ.

Before proving Theorem 2 we first establish two lemmas.

LEMMA 1. $\max_{f\in\mathcal{P}} f(A^*A)-|f(A)|^2=\min_\lambda \|A-\lambda\|^2$.

PROOF. Continuity considerations and the weak* compactness of $\mathcal{P}$ show that the words "max" and "min" are appropriate. Now if $f \in \mathcal{P}$, then

$$\begin{aligned} f((A-\lambda)^*(A-\lambda)) &- |f(A-\lambda)|^2 \\ &= f(A^*A - 2\mathrm{Re}\bar{\lambda}A + |\lambda|^2) - (|f(A)|^2 - 2\mathrm{Re}\bar{\lambda}f(A) + |\lambda|^2) \\ &= f(A^*A) - |f(A)|^2. \end{aligned}$$

Hence $f(A^*A) - |f(A)|^2 \leqq f((A-\lambda)^*(A-\lambda)) \leqq \|A-\lambda\|^2$ for all complex λ, so that $\max_f \leqq \min_\lambda$. The reverse inequality requires another lemma:

LEMMA 2. *If $f_0 \in \mathcal{P}$ satisfies $f_0(A^*A) - |f_0(A)|^2 = \max$, then*

$$|f_0(A) - g(A)|^2 \leqq (f_0(A^*A) - |f_0(A)|^2) - (g(A^*A) - |g(A)|^2)$$

for all $g \in \mathcal{P}$.

PROOF. Fix $g \in \mathcal{P}$ and let $f_t = (1-t)f_0 + tg$ for $0 \leqq t \leqq 1$. Then $f_t \in \mathcal{P}$ so that

$$f_t(A^*A) - |f_t(A)|^2 \leqq f_0(A^*A) - |f_0(A)|^2.$$

This inequality in turn gives

$$\begin{aligned} t|g(A) - f_0(A)|^2 &+ (f_0(A^*A) - |f_0(A)|^2) \\ &- (g(A^*A) - |g(A)|^2) - |f_0(A) - f(A)|^2 \geqq 0, \end{aligned}$$

and the proof is completed by letting t tend to 0.

To complete the proof of Lemma 1, choose a maximal f_0 as in Lemma 2. Then for any $g \in \mathcal{P}$

$$\begin{aligned} g((A - f_0(A))^*(A - f_0(A))) &= g(A^*A) - 2\mathrm{Re}\overline{f_0(A)}g(A) + |f_0(A)|^2 \\ &= g(A^*A) - |g(A)|^2 + |g(A) - f_0(A)|^2 \\ &\leqq f_0(A^*A) - |f_0(A)|^2, \end{aligned}$$

the last step being justified by Lemma 2. Therefore,

$$\|A - f_0(A)\|^2 = \sup_{g \in \mathcal{P}} g((A - f_0(A))^*(A - f_0(A))) \leqq f_0(A^*A) - |f_0(A)|^2.$$

This completes the proof of Lemma 1.

PROOF OF THEOREM 2. Assume that condition (i) of the theorem holds and choose a maximal $f_0 \in \mathcal{P}$ as in Lemma 2. Then

$$\|A^*A\| - |f_0(A)|^2 \geqq f_0(A^*A) - |f_0(A)|^2 = \|A\|^2 = \|A^*A\|$$

and it follows that $f_0(A) = 0$, $f_0(A^*A) = \|A\|^2$. Hence (i) implies (ii).

If condition (ii) holds, then

$$\|A\|^2 + |\lambda|^2 = f(A^*A) + |\lambda|^2 = f((A+\lambda)^*(A+\lambda)) \leqq \|A+\lambda\|^2$$

for all complex λ, and this is condition (iii).

Since it is clear that (iii) implies (i), this completes the proof of Theorem 2.

REMARKS. (1) The equivalence of conditions (i) and (iii) was first established indirectly by J. G. Stampfli [6] as a consequence of his proof that the inner derivation $X \rightarrow AX - XA$ on $\mathfrak{B}(H)$ has norm equal to $2 \inf_\lambda \|A - \lambda\|$.

(2) The inequality in (iii) cannot be improved to equality. For example, if $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ on two-dimensional Hilbert space, then

$$\|A - \lambda\| = \tfrac{1}{2}(1 + (4|\lambda|^2 + 1)^{1/2}) > (\|A\|^2 + |\lambda|^2)^{1/2}$$

for all $\lambda \neq 0$.

(3) It is a consequence of Theorem 2 that for each $A \in \mathfrak{B}(H)$ there is a unique $\lambda = \lambda(A) \in W(A)^-$ such that

$$\|A - \mu\|^2 \geqq \|A - \lambda(A)\|^2 + |\lambda(A) - \mu|^2$$

for all complex μ.

2. **Self-commutators.** A Hermitian operator A on a Hilbert space is called a self-commutator if there exists an operator T such that $A = T^*T - TT^*$. This is equivalent to the condition $A = i(BC - CB)$ with B and C Hermitian.

THEOREM 3. *The set $\mathcal{S}$ of self-commutators is norm closed in $\mathfrak{B}(H)$.*[2]

PROOF. Let $\mathcal{K}$ be the ideal of compact operators on H. The *essential numerical range* [8] of an operator B on H is the numerical range $W_0(\hat{B})$ of the coset $\hat{B}$ containing B in the quotient algebra $B(H)/\mathcal{K}$. More explicitly, $W_0(\hat{B}) = \bigcap W(B+K)^-$ where the intersection is taken over all compact operators K [8].

By a result of Radjavi [5], a Hermitian operator A belongs to $\mathcal{S}$ if and only if $0 \in W_0(\hat{A})$. Thus it suffices to prove that if $\mathfrak{B}$ is a Banach algebra with identity, then the elements $A \in \mathfrak{B}$ such that $0 \in W_0(A)$ form a closed set in $\mathfrak{B}$. The criterion in Theorem 1 immediately gives the following stronger assertion:

LEMMA 3. *If $A_n, A \in \mathfrak{B}$ and $\|A_n - A\| \rightarrow 0$, then*

$$\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} W_0(A_n) \subset W_0(A).$$

[2] The full set of commutators, however, is norm dense [1].

3. **Finite operators.** For the remainder of the paper H denotes a separable infinite dimensional complex Hilbert space. By a finite operator we shall mean a bounded linear operator A on H such that $0 \in W(AX-XA)^-$ for each $X \in \mathcal{B}(H)$.

THEOREM 4. *These are equivalent conditions on an operator A:*
(i) *A is finite.*
(ii) $\inf_X \|AX-XA-1\| = 1$.
(iii) *There exists $f \in \mathcal{P}$ such that $f(AX) = f(XA)$ for all $X \in \mathcal{B}(H)$.*

PROOF. Conditions (i) and (ii) are equivalent by Theorem 1. Conditions (ii) and (iii) are equivalent by the Hahn-Banach Theorem.

COROLLARY. *The class $\mathfrak{F}$ of finite operators is norm closed in $\mathcal{B}(H)$. Moreover, if $A \in \mathfrak{F}$, then the C^*-algebra generated by A is contained in $\mathfrak{F}$.*

PROOF. Condition (ii) of Theorem 4 clearly implies that $\mathfrak{F}$ is norm closed.

If $A \in \mathfrak{F}$, then there is a positive linear functional f such that $f(AX) = f(XA)$ for all $X \in \mathcal{B}(H)$. Let $\mathcal{B}(f) = \{B \in \mathcal{B}(H) : f(BX) = f(XB)$ for all $X \in \mathcal{B}(H)\}$. It is easy to see that $\mathcal{B}(f)$ is a C^*-algebra, and $\mathcal{B}(f) \subset \mathfrak{F}$ by Theorem 4. Since $\mathcal{B}(f)$ contains A, it follows that the C^*-algebra generated by A belongs to $\mathfrak{F}$.

Putnam [4] proved that every hyponormal operator is a finite operator. Since a hyponormal operator A has norm equal to its spectral radius $|\sigma(A)|$, this result is included in the following theorem of Halmos [2]:

THEOREM 5. *Let $\mathcal{B}$ be a complex Banach algebra with identity. If $A \in \mathcal{B}$ and $\|A\| = |\sigma(A)|$, then $\|AX-XA-1\| \geqq 1$ for all $X \in \mathcal{B}$.*

Let $\mathcal{K}$ be the ideal of compact operators on H. The canonical map $A \to \hat{A}$ from $\mathcal{B}(H)$ onto $\mathcal{B}(H)/\mathcal{K}$ is a norm-decreasing homomorphism. Hence Theorem 5 implies that any operator with the property $\|\hat{A}\| = |\sigma(\hat{A})|$ is a finite operator on H. In particular, $\mathfrak{F}$ contains every operator of the form normal+compact, or more generally, every A such that $\hat{A}$ is hyponormal. In fact, Theorem 1 allows a stronger deduction:

COROLLARY 1. *If $\|\hat{A}\| = |\sigma(\hat{A})|$, then 0 belongs to the essential numerical range of $AX-XA$ for all $X \in \mathcal{B}(H)$.*

COROLLARY 2. *If A and B are normal operators, then there exists an operator X such that $B = AX-XA$ only if 0 belongs to the convex hull of the essential spectrum of B.*

PROOF. A normal element of the C^*-algebra $\mathcal{B}(H)/\mathcal{K}$ has the property that its numerical range is the convex hull of its spectrum (see Corollary 2 of Theorem 1).

For each positive integer n let $\mathcal{R}_n$ denote the set of operators on H that have an n-dimensional reducing subspace.

THEOREM 6. *$\mathcal{R}_n^- \subset \mathfrak{F}$ for $n \geqq 1$.*

PROOF. Let

$$\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$$

be the matrix representation of A relative to the decomposition $H = H_0 \oplus H_1$, where H_0 is an n-dimensional reducing subspace of A. Then any operator X on H has a representation

$$X = \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix}.$$

A simple computation shows that $W(AX - XA)^-$ contains the numerical range of $A_0X_0 - X_0A_0$. Since H_0 is finite dimensional, the latter commutator has trace 0 and thus

$$0 = \frac{1}{n}\operatorname{Tr}(A_0X_0 - X_0A_0) \in W(A_0X_0 - X_0A_0)^-$$

since the numerical range is a convex set whose closure contains the spectrum.

If $A = X \oplus F$ where F is an operator of finite rank, then the vector sum of the ranges of F and F^* is a finite dimensional reducing subspace for A, hence $A \in \bigcup_n \mathcal{R}_n$. Hence $\mathfrak{F}$ contains every operator that can be written as a uniform limit of operators each having a summand of finite rank. In particular, $\mathfrak{F}$ contains every operator with a compact direct summand. (However, $\mathfrak{F}$ is not invariant under compact perturbations.)

REMARKS. (1) Theorem 6 is of interest in connection with an (open) question of Halmos [3] that asks whether the set of reducible operators is dense in $\mathcal{B}$(H). Our result implies that the "finitely reducible" operators are not dense.

(2) The set $\mathcal{R}_1^-$ is larger than it may appear at first sight. Thus Stampfli [7] has shown that each of the following is a sufficient condition for an operator A to belong to $\mathcal{R}_1^-$:

(i) $\|A - \lambda\| = |\sigma(A - \lambda)|$ for some complex number λ.

(ii) $A = H + K$, where H is hyponormal and K is compact.

(iii) $A = T+K$, where T is a Toeplitz operator and K is compact.

$\mathcal{R}_1^-$ also contains certain quasi-nilpotent operators. For example, if T is quasi-nilpotent and Re $T \geqq 0$, then $(I+T)^{-1}$ has equal norm and spectral radius. Hence $(1+T)^{-1} \in \mathcal{R}_1^-$ and this implies that T belongs to $\mathcal{R}_1^-$.

THEOREM 7. *If A satisfies a quadratic polynomial, then $A \in \mathfrak{F}$.*

PROOF. If A does not belong to $\mathfrak{F}$, then there is an operator X such that $W(AX-XA)^-$ lies in the open right half-plane. Consider the operator J defined on $\mathcal{B}(H)$ by $J(Z) = (AX-XA)Z+Z(AX-XA)$. The spectrum of J is contained in the algebraic sum $\sigma(AX-XA) +\sigma(AX-XA)$, and therefore is contained in the open right half-plane. In particular, J is invertible. However, if $(A-\alpha)(A-\beta) = 0$, then $J(A-\frac{1}{2}(\alpha+\beta)) = 0$.

THEOREM 8. *The nilpotent*

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ A & 1 & 0 \end{pmatrix}$$

is a finite operator on $H^{(3)} = H \oplus H \oplus H$ if and only if A is a finite operator on H.

PROOF. A state f on $\mathcal{B}(H^{(3)})$ has the property $f(TX) = f(XT)$ for all $X \in \mathcal{B}(H^{(3)})$ if and only if f has the form $f(X) = \frac{1}{3}g(X_{11}+X_{22}+X_{33})$, where the X_{ii} are the diagonal entries of X, and g is a state on $\mathcal{B}(H)$ satisfying $g(AX) = g(XA)$ for all $X \in \mathcal{B}(H)$. (This assertion is trivial in one direction; the other follows from the facts that f is selfadjoint and $f(T^2X) = f(XT^2)$ for all X by the Corollary of Theorem 4.)

COROLLARY. (1) *The von Neumann algebra generated by a finite operator need not be of finite type.*

(2) *The class $\mathfrak{F}$ is not invariant under similarity transformation.*

(3) $(\bigcup_n \mathcal{R}_n)^-$ *does not contain every nilpotent operator.*

PROOF. The second assertion is a consequence of the fact that the nilpotents in Theorem 8 are all similar to the one constructed with $A = 0$. To prove the first assertion, choose an irreducible finite operator for A (e.g., A = simple unilateral shift); the corresponding T is irreducible and finite.

4. **Open questions.** The adjective "finite" used to describe the operators in $\mathfrak{F}$ is admittedly ad hoc. Justification of the term would

seem to require answers to the following questions:

(1) Is $\bigcup_n \mathcal{R}_n^-$ dense in $\mathfrak{F}$?

(2) Is every operator similar to a finite operator?

(3) For which finite operators A is every similarity transformation $S^{-1}AS$ also finite?

An operator A belongs to $\mathfrak{F}$ precisely when the inner derivation $\mathrm{ad}(A): X \to AX - XA$ has range orthogonal to 1. In particular, the range of $\mathrm{ad}(A)$ is not dense.

(4) Does there exist an operator A such that the range of $\mathrm{ad}(A)$ is dense in $\mathcal{B}(H)$?

It follows from Theorem 4 that the C^*-algebra $\mathcal{A}$ generated by a finite operator admits a representation whose weak closure is a finite von Neumann algebra. What else can one say about $\mathcal{A}$? In particular, the referee asks for a description of the traces on $\mathcal{A}$ (is there a faithful one, do they separate points of $\mathcal{A}$, what are the extreme traces?).

In connection with questions (1) and (4), one sees easily that $A \in \bigcup_n \mathcal{R}_n$ if and only if the set $[\mathcal{A}, \mathcal{B}(H)]$ of commutators is not weak*-dense in $\mathcal{B}(H)$.

References

1. A. Brown and C. Pearcy, *Structure of commutators of operators*, Ann. of Math. (2) **82** (1965), 112–127. MR **31** #2612.

2. P. R. Halmos, *Commutators of operators*. II, Amer. J. Math. **76** (1954), 191–198 MR **15**, 538.

3. ——— *Irreducible operators*, Michigan Math. J. **15** (1968), 215–223. MR **37** #6788.

4. C. R. Putnam, *On commutators of bounded matrices*, Amer. J. Math. **73** (1951), 127–131. MR **12**, 836.

5. H. Radjavi, *Structure of $A^*A - AA^*$*, J. Math. Mech. **16** (1966), 19–26. MR **34** #3332.

6. J. G. Stampfli, *The norm of a derivation*, Pacific J. Math (to appear).

7. ———, *On hyponormal and Toeplitz operators*, Math. Ann. **183** (1969), 328–336.

8. J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tôhoku Math. J. **20** (1968), 417–424.

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