GRAPHS WITH A LARGE CAPACITY

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Abstract. A constructive method for obtaining graphs with a relatively large capacity is given. The method uses products of graphs.

Introduction. In this note we present a constructive method for obtaining graphs with a relatively large capacity and obtain an upper bound for the capacity $\theta(G)$ of a graph $G$. The capacity of a graph was introduced by Shannon [5], for investigations of problems concerning noisy channels in information theory. If $\mu(G)$ is the maximal number of independent vertices in the graph $G$, it is well known [5], that $\theta(G) \geq \mu(G)$. Our method will yield for every $k \geq 0$, a graph $G_k$ such that $\theta(G_k) \geq k \cdot \mu(G_k)$. The construction is based on the composition of graphs introduced by Harary [1].

The definitions and generally accepted notations throughout this paper will be those used in Harary [2]. $\mu(G)$ will denote the maximal number of independent vertices in $G$. Since the strong product and composition of graphs (for definitions see Harary [1]) are associative, powers of $G$ with respect to each one of them is well defined. We denote those powers by $G^n$ and $G^{[n]}$ resp. The capacity of $G$ is defined by

$$\theta(G) = \sup_n \mu(G^n)^{1/n}.$$

Upper bound for $\theta(G)$: If $A$ and $B$ are independent sets in $G$ and $H$ resp., $A \times B$ is independent in $G \times H$, hence

$$\mu(G \times H) \geq \mu(G) \cdot \mu(H) \Rightarrow \mu(G^n) \geq \mu^n(G) \Rightarrow \theta(G) \geq \mu(G).$$

To obtain an upper bound, we consider the function $\alpha(G)$ introduced in [4], and defined as follows: Let $V(G) = \{g_1, \cdots, g_n\} \cdot \{C_1, \cdots, C_s\}$ is a fixed ordering of all maximal complete subgraphs of $G$. $\alpha'_t = 1$ if $g_i \in C_j$, $\alpha'_t = 0$ otherwise

$$P(G) = \left\{(x_1, \cdots, x_n) \mid x_i \geq 0, \sum_{i=1}^{n} \alpha'_t x_i \leq 1, 1 \leq j \leq s \right\},$$

$$\alpha(G) = \max_{x \in P(G)} \sum_{i=1}^{n} x_i, \quad x = (x_1, \cdots, x_n).$$

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Theorem 1. If $G$ is an ordinary graph then
$$\theta(G) \leq \alpha(G).$$

Proof. Let $H$ be any graph and let $A \subset V(G \times H)$ be a maximal independent set of vertices in $G \times H$ (card $A = \mu(G \times H)$). Let
$$A_i = \{h \mid (g, h) \in A\}, \quad A'_i = \{(g, h) \mid (g, h) \in A\}.$$  
Obviously, $\{A'_i\}$ is a disjoint decomposition of $A$. Hence
$$\sum_{i=1}^{n} \text{card } A'_i = \mu(G \times H).$$
By the definition of $A_i$ and the independence of $A$ we have $\text{card } A_i = \text{card } A'_i$, $A_i$ is an independent set in $H$. Let
$$x_i = \text{card } A_i / \mu(H).$$
We will show first that $(x_1, \cdots, x_n) \in P(G)$. For simplicity of notation, we may assume without loss of generality, that $C = \{g_1, \cdots, g_k\}$.
Since $A$ is independent, and $(g_i, g_j) \in E(G)$, $1 \leq i, j \leq k$, we must have $A_i \cap A_j = \emptyset$ and $A_i \cup A_j$ is an independent set in $H$. By the same argument, $\bigcup_{i=1}^{k} A_i$ is an independent set in $H$ and the union is disjoint. Hence
$$\mu(H) \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{k} \text{card } A_i = \text{card } \bigcup_{i=1}^{k} A_i \leq \mu(H) \Rightarrow \sum_{i=1}^{n} \alpha_i x_i \leq 1.$$  
Therefore $(x_1, \cdots, x_n) \in P(G)$. Hence we have
$$\alpha(G) \geq \sum_{i=1}^{n} x_i = \frac{1}{\mu(H)} \sum_{i=1}^{n} \text{card } A_i = \frac{1}{\mu(H)} \sum_{i=1}^{n} \text{card } A'_i = \frac{\mu(G \times H)}{\mu(H)}$$  
$$\Rightarrow \quad \alpha(G) \cdot \mu(H) \geq \mu(G \times H).$$
Since $\alpha(G) \geq \mu(G)$ [4], by induction we obtain
$$\mu(G^n) \leq \mu(G^{n-1}) \alpha(G) \leq \alpha^n(G) \Rightarrow \theta(G) \leq \alpha(G).$$

Remark. In [4], it was shown that $\mu(G \times H) = \mu(G) \cdot \mu(H)$ for all graphs $H$ iff $\alpha(G) = \mu(G)$. Hence for such graphs we have $\theta(G) = \alpha(G) = \mu(G)$. Since the capacity of no other graphs is known nothing else can be said about the upper bound established above.

Theorem 2. For every $k > 0$, there exists a graph $G_k$ such that $\theta(G_k) \geq k \mu(G_k)$.  

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Proof. Let \( G_0 \) be a self-complemented graph with \( n \) vertices such that \( \mu(G_0) \cdot \mu(\overline{G_0}) < n \) (e.g. a pentagon). Let \( p \) be a positive integer such that \( n^p \leq k^p \mu^p(G_0) \). Obviously, such a number exists since by our assumptions \( \mu^2(G_0) = \mu(G_0) \cdot \mu(\overline{G_0}) < n \). Let \( G_k = G_0^{[k]} \). It is easy to see that \( \mu(G[H]) = \mu(G) \cdot \mu(H) \), hence \( \mu(G_k) = \mu^p(G_0) \). If \( G \) and \( H \) are self-complemented, it was shown by Sabidussi that \( G[H] \) is self-complemented, hence \( G_k \) is self-complemented. Consider the set \( A = \{(g, g) \mid g \in V(G_k)\} \) as a subset of \( V(G_k \times \overline{G_k}) \). Since \((g, g') \in E(G_k) \Rightarrow (g, g') \in E(\overline{G_k}) \) it follows that \( A \) is an independent set of vertices in \( G_k \times \overline{G_k} \). Since \( \text{card } A = n^p \) we get

\[
\mu(G_k^2) = \mu(G_k \times \overline{G_k}) \geq n^p.
\]

Ljubić [3] has shown that \( \theta(G) = \lim_n (\mu(G^n))^{1/n} \) therefore we have in general

\[
\theta(G) = \lim_n (\mu(G^{2n}))^{1/2n} = \left( \lim_n \mu(G^{2n})^{1/n} \right)^{1/2} = \theta^{1/2}(G^2).
\]

By using the lower bound for the capacity of a graph and (*) we obtain

\[
\theta(G_k) = \theta^{1/2}(G_k^2) \geq n^{p/2} \geq k \mu^p(G_0) = k \mu(G_0^{[k]}) = k \cdot \mu(G_k).
\]

References