RECOGNIZING CERTAIN FACTORS OF E^4

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ABSTRACT. It has been proved that for certain peculiar decomposition spaces Y of euclidean 3-space E^3, $Y \times E^1$ is homeomorphic to euclidean 4-space, E^4. In this paper we prove that if a decomposition space Y of E^3 is generated by a trivial defining sequence whose elements are cubes with handles, and this sequence can be replaced by a toroidal defining sequence, then $Y \times E^1$ is homeomorphic to E^4.

For each natural number i, let A_i be a disjoint, locally finite set of cubes with handles imbedded in E^3; let $A_i^* = \bigcup \{a \mid a \in A_i\}$. The components of $X = \bigcap A_i^*$ are the nondegenerate elements of an upper semicontinuous decomposition $G = G(\{A_i\})$ of E^3 and $\{A_i\}$ will be called a *defining sequence* for G. If $G(\{B_i\}) = G(\{A_i\})$ we shall say $\{A_i\}$ can be *replaced by* $\{B_i\}$. In [**1**] the authors conjectured that if the defining sequence $\{A_i\}$ is trivial, then E^3/G is a factor of E^4. Theorem 2 below is a partial solution to that conjecture.

If each A_i is a set of solid tori, then we say $\{A_i\}$ is *toroidal*. It is our contention that if the defining sequence $\{A_i\}$ is trivial and can be replaced by a toroidal defining sequence $\{B_i\}$, then E^3/G is a factor of E^4. The main distinction to be made here is that $\{B_i\}$ need not be trivial. For related results see [**2**], [**3**], and [**4**].

A close examination of the proof of Theorem 2 of [**1**] will show that the requirement that $\{A_i\}$ be trivial, i.e., that each inclusion $j: A_{i+1}^* \subset A_i^*$ be null homotopic could be replaced by the requirement that for each i, $j: X \subset A_i^*$ be null homotopic. This is stated in the following theorem.

THEOREM 1. *If $\{A_i\}$ is a toroidal defining sequence for G, and for each i, the inclusion $j: X \subset A_1^*$ is null homotopic, then E^3/G is a factor of E^4.*

It is easy to show that if a trivial defining sequence $\{A_i\}$ can be replaced by $\{B_i\}$, then for each i, the inclusion $j: X \subset B_i^*$ is null homotopic. This can be seen by observing that if $T \in B_i$, then $T \cap X \subset \operatorname{Int}(T)$, $T \cap X$ is compact, and thus for some k there exists a finite set $\{S_1, \cdots, S_m\} \subset A_k$ such that $T \cap X \subset \bigcup S_i \subset T$. Then since X is null homotopic in A_k^* and the S_i are components of A_k^*,

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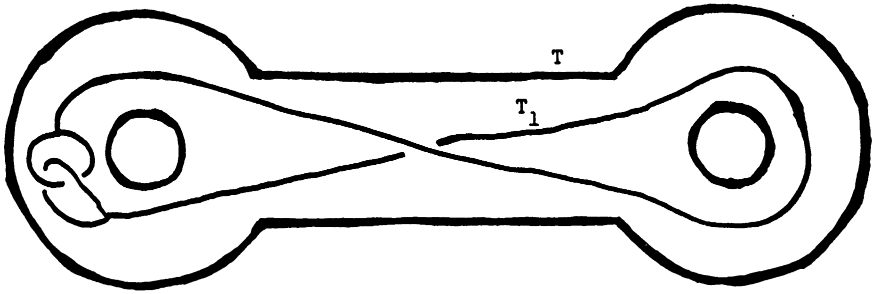
$T \cap X$ is null homotopic in $\bigcup S_i$ and hence in T. We therefore obtain the main result.

THEOREM 2. *If* $\{A_i\}$ *is a trivial defining sequence for* G *and* $\{A_i\}$ *can be replaced by a toroidal defining sequence, then* E^3/G *is a factor of* E^4.



FIGURE

If G is the decomposition generated by 2-holed solid tori, as in the Figure, it can easily be seen that by filling in the holes of T_1 with 3-cells, a solid torus containing T_1 but contained in T can be constructed. Then the original defining sequence can be replaced by a toroidal defining sequence which is not trivial. Nevertheless, by Theorem 2, E^3/G is a factor of E^4.

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