GENERALIZED BALAYAGE AND A RADON-NIKODYM THEOREM

D. J. HEBERT, JR.[1]

ABSTRACT. A simplified proof is given of Doob's result that a balayage ordered collection of probability measures on a compact Hausdorff space K yields a K-valued supermartingale with the measures as marginal distributions. The proof shows further connections with martingale convergence theory.

The main theorem of a recent article by Doob [1] says that a balayage ordered collection of probability measures on a compact Hausdorff space K yields a K-valued supermartingale with the given measures as marginal distributions. The theorem requires no metrizability assumptions, but its proof uses a theorem of Meyer on dilations of measures on a compact metrizable space [3, p. 246]. The purpose of this note is to give a simple proof of a version of Doob's theorem avoiding metrizability arguments. Meyer's theorem is an easy corollary. A Radon-Nikodym-Metivier type lemma makes use of the Ionescu Tulcea lifting theorem and Helms' martingale convergence theorem (see [3]), both of which are consequences of Doob's original martingale theory.

Let K be a compact Hausdorff topological space, let $C(K)$ be the Banach lattice of continuous real valued functions on K with the supremum norm, and let $C(K)^*$ be the Banach lattice dual of $C(K)$. The space $C(K)^*$ is identified with the space of all signed regular Borel measures on K, and for f in $C(K)$ and λ in $C(K)^*$ the notation $\lambda(f)=\langle\lambda, f\rangle=\int f d\lambda$ is used interchangeably. The set of positive elements of norm 1 in $C(K)^*$ is denoted by $P(K)$. A mapping T of K into $C(K)^*$ is said to be weakly λ-measurable whenever the function $x\rightarrow\langle T(x), f\rangle$ is λ-measurable for each f in $C(K)$. The following lemma is a variation of a Radon-Nikodym theorem of Metivier [2].

LEMMA. *Suppose that λ is in $P(K)$ and $\mathfrak{F}$ is the σ-field of λ-measurable sets. If m is an additive function from $\mathfrak{F}$ into the positive cone of $C(K)^*$ such that $|\langle m(E), f\rangle| \leq \|f\|\cdot\lambda(E)$ and $\langle m(K), 1\rangle=1$, then there is a*

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weakly λ-measurable function T from K into $P(K)$ such that for each E in $\mathfrak{F}$ and for each f in $C(K)$,

$$\langle m(E), f\rangle = \int_E \langle T(x), f\rangle \lambda(dx).$$

PROOF. For each f in $C(K)$, for each x in K, and for each finite partition π of $\mathfrak{F}$ define $T(\pi, x, f) = \langle m(E), f\rangle/\lambda(E)$ where E is the member of π containing x. If $\mathfrak{F}_\pi$ denotes the Boolean algebra generated by π, the collection $\{T(\pi, x, f), \mathfrak{F}_\pi\}$ forms a bounded martingale for each f. By a theorem of Helms (see [3, p. 86]), the martingale converges in $L^1(K, \mathfrak{F}, \lambda)$ to an integrable function T_f which is clearly bounded by $\|f\|$. The mapping $f \to T_f$ is a bounded linear mapping of $C(K)$ into $L^\infty(K, \mathfrak{F}, \lambda)$. Let ρ be a lifting of $L^\infty(K, \mathfrak{F}, \lambda)$. The mapping T defined by $\langle T(x), f\rangle = [\rho(T_f)](x)$, for each f and x, is the desired function.

THEOREM. *Let S be a cone in $C(K)$ closed under the lattice operation $\wedge$ and containing the constant functions. If μ and λ are members of $P(K)$ such that $\mu(h) \leqq \lambda(h)$ for each h in S, then there is a weakly λ-measurable mapping T from K into $P(K)$ such that*

$$\mu(f) = \int_K \langle T(x), f\rangle \lambda(dx) \quad \textit{for } f \textit{ in } C(K),$$

and

$$\int_E \langle T(x), h\rangle \lambda(dx) \leqq \int_E h(x)\lambda(dx)$$

for each h in S and for each λ-measurable set E.

PROOF. For each f in $C(K)$ let $f^*(x) = \inf\{h(x): h \geqq f,\ h \in S\}$. Properties of this function are given in [3]. Among these are the following: the equation $s(f) = \int f^*(x)\lambda(dx)$ defines a sublinear functional s on $C(K)$, and since λ dominates μ on S, s dominates μ on $C(K)$. Let B be the normed linear space of equivalence classes of λ-measurable simple functions from K to $C(K)$. The space $C(K)$ can be identified with the λ-almost everywhere constant functions. The functional s can be extended to a sublinear functional s^* on B defined by the equation $s^*(g) = \int [g(x)]^*(x)\lambda(dx)$. By the Hahn-Banach theorem there is a linear functional μ^* dominated by s^* which extends μ to all of B. For a subset E of K and for f in $C(K)$ let Ef denote the function whose values are f on E and 0 outside E. Thus $\mu^*(Ef) \leqq s^*(Ef) = \int_E f^*(x)\lambda(dx)$ for f in $C(K)$ and for λ-measurable E.

The function m defined by the equation $\langle m(E), f\rangle = \mu^*(Ef)$ satisfies the hypotheses of the lemma. The resulting function T is the one required to conclude the proof.

The function T considered as a transition probability on K can be used to define a product measure P on $K_1 \times K_2 = K \times K$. If $f(x, y)$ is a continuous function on $K \times K$ and $f_x(y) = f(x,y)$, then $P(f) = \int \langle T(x), f_x\rangle \lambda(dx)$, so that λ and μ are the natural projections of P onto K_1 and K_2 respectively. Further, if X_i is the ith coordinate function of a point X in $K_1 \times K_2$ and E is a Borel subset of K then

$$\int_{E\times K} h(X_1)dP = \int_E h(x)\lambda(dx) \geqq \int \langle T(x), h\rangle \lambda(dx) = \int_{E\times K} h(X_2)dP.$$

Thus if $\mathfrak{F}_2$ is the σ-field of Borel sets of $K \times K$ and $\mathfrak{F}_1$ is composed of sets of the form $E \times K$ where E is a Borel subset of K, then $\{(h(X_i), \mathfrak{F}_i): i = 1, 2\}$ is a supermartingale. This argument is easily extended to finite and then to arbitrary index sets to produce the following

COROLLARY (DOOB). *Suppose that I is an ordered set and $\{\lambda_i : i \in I\}$ is a collection of members of $P(K)$ such that $i \leqq j$ if and only if $\lambda_i(h) \geqq \lambda_j(h)$ for each h in S. There is a probability space $(\Omega, \mathfrak{F}, P)$, a family $\{X_i : i \in I\}$ of $\mathfrak{F}$-measurable functions, and a family $\{\mathfrak{F}_i : i \in I\}$ of sub-σ-fields of $\mathfrak{F}$ such that $\{(h(X_i), \mathfrak{F}_i) : i \in I\}$ is a supermartingale for each h in S and λ_i is the distribution of X_i for each i in I.*

If K is metrizable, then T can be redefined as a Borel measurable function such that $\langle T(x), h\rangle \geqq h(x)$ for each x in K and h in S. This is Meyer's theorem.

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UNIVERSITY OF PITTSBURGH, PITTSBURGH, PENNSYLVANIA 15213