TREE-LIKE CONTINUA AND CELLULARITY

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Abstract. In this paper the equivalence of tree-like and cellular is proved for 1-dimensional continua in $E^n$. More precisely, if $X$ is a tree-like continuum, then the collection of all embeddings $h: X \to E^n$, $n \geq 3$, such that $h[X]$ is cellular in $E^n$ is a dense $G_\delta$-subset of the collection of all maps from $X$ into $E^n$. Conversely, if $X$ is a 1-dimensional cellular subset of $E^n$, then $X$ is a tree-like continuum.

1. Terminology. Throughout this paper a continuum will be a nondegenerate compact connected metric space and a covering will be a finite open covering. The symbol $\sim$ should be translated "homotopic to." If $X$ is a continuum and $\mathcal{O} = \{O_1, \ldots, O_m\}$ is a covering of $X$, the mesh of $\mathcal{O}$, denoted $\text{mesh } \mathcal{O}$, is the maximum of the diameters of the elements of $\mathcal{O}$. The nerve of $\mathcal{O}$, denoted $\mathcal{N}(\mathcal{O})$, is the abstract complex consisting of those simplexes $(O_{i_1} \cdots O_{i_j})$ such that $O_{i_1} \cap \cdots \cap O_{i_j} \neq \emptyset$. A continuum $X$ is tree-like if for each $\varepsilon > 0$ there exists a covering $\mathcal{O}$ of $X$ such that mesh $\mathcal{O} < \varepsilon$ and $\mathcal{N}(\mathcal{O})$ is a contractible 1-complex.

Let $X$ be a subset of a topological space $Y$ and let $n$ be a nonnegative integer. The statement that $X$ has property $n$-UV means that for each open set $U$ containing $X$, there is an open set $V$ containing $X$ and contained in $U$ such that each singular $n$-sphere in $V$ is homotopic to 0 in $U$. $X$ has property $UV^n$ if it has property $i$-UV for each $i \leq n$ and $X$ has property $UV^\infty$ if it has property $i$-UV for each nonnegative integer $i$. $X$ has property $UV^\infty$ if for each open set $U$ containing $X$, there is an open set $V$ containing $X$ and contained in $U$ such that $V$ is contractible in $U$. For a good discussion of the UV properties the reader is referred to Armentrout [1].

A subset $X$ of $E^n$ is said to be cellular in $E^n$ if there is a sequence $C_1, C_2, \cdots$ of $n$-cells in $E^n$ such that

1. for each positive integer $i$, $C_{i+1} \subset \text{Int } C_i$, and
2. $\bigcap_{i=1}^\infty C_i = X$.

This paper is devoted to studying the relationship between tree-like, the UV-properties, and cellularity in Euclidean space. In §2 we show that for 1-dimensional continua they are essentially the same and in §3 we prove an embedding theorem for tree-like continua.

Received by the editors December 8, 1969.

AMS 1970 subject classifications. Primary 5450; Secondary 5420, 5425.

Key words and phrases. Cellularity, continua, dimension, tree-like, UV-properties.
2. An equivalence theorem. In this section we shall show that a 1-dimensional continuum $X$ is tree-like if and only if the image of each embedding of $X$ into $E^n$ has property $UV^\infty$. This is equivalent to the statement that there is an embedding $h$ of $X$ into some Euclidean space such that $h[X]$ is cellular.

**Lemma 1.** Let $X$ be a continuum in $E^n$, $n \geq 3$. If $X$ is 1-dimensional, then $X$ has property $i$-UV for $i = 0, 2, 3, \ldots$. If $X$ is tree-like, then $X$ has property $UV^\infty$.

**Proof.** Let $U$ and $W$ be open subsets of $E^n$ such that $\overline{W}$ is compact and $X \subset W \subset \overline{W} \subset U$. There is a positive real number $\epsilon$ such that if $A$ is any subset of $U$ which meets $\overline{W}$ and has diameter less than $\epsilon$, then the convex hull of $A$ is contained in $U$.

Let $\mathcal{O} = \{O_1, \ldots, O_m\}$ be a covering of $X$ by open sets contained in $W$ such that mesh $\mathcal{O} < \epsilon/3$ and $\mathfrak{N}(\mathcal{O})$ is a 1-complex. If $X$ is tree-like then $\mathcal{O}$ may be chosen so that $\mathfrak{N}(\mathcal{O})$ is contractible. For each $i = 1, \ldots, m$, let $p_i$ be a point of $O_i$ such that the set $\{p_1, \ldots, p_m\}$ is in general position in $E^n$. Since $n \geq 3$, the collection $L$ consisting of vertices $p_1, \ldots, p_m$ and 1-simplexes $(p_ip_j)$ such that $O_i \cap O_j \neq \emptyset$ is a subcomplex of $E^n$. The choice of $\epsilon$ implies that $L$ is contained in $U$. Moreover, $L$ is the image under a simplicial embedding of $\mathfrak{N}(\mathcal{O})$ into $E^n$ and therefore is contractible if $X$ is tree-like.

Let $V = \bigcup_{i=1}^m O_i$. Using the methods employed in [4, p. 69], there is a mapping $f$ from $V$ onto $L$ such that $O_i = f^{-1}[s^0 p_i]$ (here $s^0 p_i$ denotes the open star of $p_i$ in $L$). Note that $f$ moves no point $x$ in $V$ more than $\epsilon$, for if $x \in O_i$, then $d(x, f(x)) \leq d(x, p_i) + d(p_i, f(x)) < \epsilon/3 + 2\epsilon/3 = \epsilon$.

Now let $S$ denote the standard $k$-dimensional sphere for some nonnegative integer $k$ and let $g : S \to V$ be a map. Then $fg$ maps $S$ into $L \subset U$ and $d(g(y), fg(y)) < \epsilon$ for each $y \in S$. Thus $fg$ and $g$ are homotopic in $E^n$ by a homotopy which moves $fg(y)$ to $g(y)$ along a straight line segment of length less than $\epsilon$. In particular, the choice of $\epsilon$ implies that $fg$ and $g$ are homotopic in $U$. But $fg[S]$ is contained in $L$ and therefore, if $k \neq 1$, $fg \sim 0$ in $L \subset U$. If $X$ is tree-like, then $fg \sim 0$ in $L \subset U$ for all nonnegative integers $k$. Thus $g \sim fg \sim 0$ in $U$ for the desired cases.

The next lemma is proved by Case and Chamberlin in [2].

**Lemma 2.** A 1-dimensional continuum is tree-like if and only if each continuous map of $X$ into any linear graph is homotopic to 0.

**Lemma 3.** If $X$ is a 1-dimensional continuum in $E^n$ having property $UV^\infty$, then $X$ is tree-like.
Proof. Let \( g: X \to K \) be a map from \( X \) into a linear graph \( K \). Since \( g \) is homotopic to a map from \( X \) onto a subcomplex of \( K \), there is no loss of generality in assuming that \( g \) is onto. Let \( p_1, \ldots, p_m \) be the vertices of \( K \) and for each \( i = 1, \ldots, m \), let \( O_i = g^{-1}[s^0 \cup p_i] \). Then \( \emptyset = \{ O_i \} \) is a covering of \( X \) and \( \mathcal{H}(\emptyset) \) is a 1-complex simplicially isomorphic to \( K \). Let \( U_1, \ldots, U_m \) be open subsets of \( E^n \) such that \( U_i \cap X = O_i \) for \( i = 1, \ldots, m \) and such that if \( \mathcal{U} = \{ U_i \} \), then \( \mathcal{H}(\mathcal{U}) \) is simplicially isomorphic to \( \mathcal{H}(\emptyset) \). Let \( U = \bigcup_{i=1}^m U_i \) and let \( f: U \to K \) be a map such that \( f^{-1}[s^0 \cup p_i] = U_i \) for \( i = 1, \ldots, m \). Note that for each \( x \in X \), \( f(x) \) and \( g(x) \) lie in the same simplex of \( K \) and therefore \( g \sim f \mid X \) in \( K \). We show \( f \mid X \) is homotopic to \( 0 \) in \( K \).

Now \( X \) has property \( UV^\infty \) in \( E^n \) and \( U \) is an open set containing \( X \), so there is a homotopy \( H': X \times [0, 1] \to U \) such that \( H'(x, 0) = x \) and \( H'(x, 1) = x_0 \) for some point \( x_0 \in U \). Define \( H: X \times [0, 1] \to K \) by \( H = f \circ H' \). Then \( H(x, 0) = f(x) \) and \( H(x, 1) = f(x_0) \).

Theorem 1. If \( X \) is a 1-dimensional continuum, then the following are equivalent:

1. \( X \) is tree-like,
2. the image of each embedding of \( X \) into \( E^n \) has property \( 1-UV \),
3. the image of each embedding of \( X \) into \( E^n \) has property \( UV^\infty \), and
4. \( X \) can be embedded as a cellular subset of some Euclidean space.

Proof. If \( n \geq 3 \), then the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) follow directly from Lemma 1. If \( n < 3 \), then Lemma 5.1 of \([1]\) applies.

If \( h: X \to E^n \) is an embedding such that \( h[X] \) has property \( UV^\infty \), then McMillan \([5]\) has shown that \( h[X] \) is cellular in \( E^{n+1} \). Thus (3) and (4) are equivalent (observing Lemma 5.1 of \([1]\) again). The proof is then completed by applying Lemma 3.

The previous theorem and the results of \([2]\) provide an interesting example concerning the \( UV \)-properties. Case and Chamberlin construct an example of a subset \( X \) of \( E^3 \) which is not tree-like, but which has trivial Čech groups.

Corollary 1. There is a 1-dimensional continuum \( X \) in \( E^3 \) which has trivial Čech homology groups, cohomology groups, and fundamental group, but not having property \( UV^\infty \) in \( E^3 \).

2. Embeddings of tree-like continua in \( E^n \). Throughout this section let \( X \) be a fixed tree-like continuum. Let \( F[X] \) denote the collection of all mappings from \( X \) into \( E^n \) with the compact open topology. Recall that \( F[X] \) is a complete metric space (cf. \([4]\)) with the usual sup metric.

Consider the following subsets of \( F[X] \):
\[ I[X] = \{ f \in F[X] | f \text{ is an embedding} \}, \]
\[ F_c[X] = \{ f \in F[X] | f[X] \text{ is cellular in } E^n \}, \]
\[ I_c[X] = F_c[X] \cap I[X]. \]

In this section we prove that if \( n \geq 3 \), then \( I_c[X] \) is a dense \( G_\delta \)-subset of \( F[X] \). Note that if \( n < 3 \), then \( I_c[X] = I[X] \). We assume therefore that \( n \) is a fixed integer \( \geq 3 \).

If \( \epsilon \) is a positive real number, an \( \epsilon \)-mapping \( f : X \rightarrow E^n \) is an element of \( F[X] \) such that for each \( y \in f[X] \), the set \( f^{-1}(y) \) has diameter less than \( \epsilon \). For each \( i = 1, 2, \ldots \), let \( G_i \) be the subset of \( F[X] \) consisting of all \( 1/i \)-mappings. The following result is proved in [4].

**Lemma 4.** For each positive integer \( i \), \( G_i \) is a dense open subset of \( F[X] \). Moreover, \( I[X] = \cap_{i=1}^{\infty} G_i \) is a dense \( G_\delta \)-subset of \( F[X] \).

For each \( i = 1, 2, \ldots \), let \( C_i \) be the collection of all elements \( f \) of \( F[X] \) such that there is an \( n \)-cell \( C \) in \( E^n \) with \( f[X] \subset \text{Int} \ C \subset N(f[X], 1/i) \). (Here, the set \( N(f[X], 1/i) \) denotes the \( 1/i \)-neighborhood of \( f[X] \) in \( E^n \).) Clearly \( F_i[X] = \cap_{i=1}^{\infty} C_i \). The following two lemmas show that each \( C_i \) is a dense open subset of \( F[X] \).

**Lemma 5.** \( F_c[X] \) is dense in \( F[X] \).

**Proof.** Let \( g \) be an element of \( F[X] \) and let \( \epsilon \) be a positive real number. Lemma 4 implies that \( X \) can be considered a subset of \( E^n \) such that \( g \) moves no point more than \( \epsilon/2 \). Corresponding to \( \epsilon/2 \), let \( f \) and \( L \) be as in the proof of Lemma 1; that is, \( f \) maps \( X \) onto the contractible 1-complex \( L \) in \( E^n \) without moving points more than \( \epsilon/2 \). Then \( f \) and \( g \) are within \( \epsilon \) of each other and, since \( L \) is collapsible, \( f[X] \) is cellular in \( E^n \).

**Lemma 6.** For each positive integer \( i \), \( C_i \) is an open subset of \( F[X] \).

**Proof.** Suppose \( f \in C_i \) and let \( C \) be an \( n \)-cell in \( E^n \) such that \( f[X] \subset \text{Int} \ C \subset N(f[X], 1/i) \). Let
\[ \epsilon = \min \{ d(f[X], E^n - \text{Int} \ C), d(C, E^n - N(f[X], 1/i)) \}. \]

Since \( \epsilon < 1/i \), any \( \epsilon/2 \)-approximation \( g \) to \( f \) will have the property that \( g[X] \subset \text{Int} \ C \subset N(g[X], 1/i) \).

**Theorem 2.** If \( n \geq 3 \), \( I_c[X] \) is a dense \( G_\delta \)-subset of \( F[X] \).

**Proof.** The previous lemmas imply that for each \( i = 1, 2, \ldots \), both \( G_i \) and \( C_i \) are dense and open in \( F[X] \). Thus \( G_i \cap C_i \) is dense and open. By Theorem 2–79 of [3], the set \( \cap_{i=1}^{\infty} (G_i \cap C_i) = I[X] \cap F_c[X] = I_c[X] \) is a dense \( G_\delta \)-subset of \( F[X] \).
The following corollary is now obvious.

**Corollary 2.** Let $X$ be a 1-dimension continuum in $E^n$ having property $UV^\infty$ and let $\epsilon$ be a positive real number. Then there is an embedding $h : X \to E^n$ such that $d(x, h(x)) < \epsilon$ for each $x \in X$ and such that $h[X]$ is cellular in $E^n$.

**References**


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