COUNTABLE CONNECTED SPACES

GARY GLENN MILLER

Abstract. Two pathological countable topological spaces are constructed. Each is quasimetrizable and has a simple explicit quasimetric. One is a locally connected Hausdorff space and is an extension of the rationals. The other is a connected space which becomes totally disconnected upon the removal of a single point. This space satisfies the Urysohn separation property—a property between $T_1$ and $T_2$—and is an extension of the space of rational points in the plane. Both are one dimensional in the Menger-Urysohn [inductive] sense and infinite dimensional in the Lebesgue [covering] sense.

1. Introduction. The first example of a countable connected Hausdorff space was given by Urysohn [18]. Other examples have been given by Hewitt [7], Bing [1], Brown [2], Golomb [6], Martin [10], Roy [15], Kirch [8], Stone [17], and Miller and Pearson [13].

A connected space $X$ has a dispersion point $x$ provided $X - x$ is totally disconnected. $X$ is a Urysohn space if for all distinct $p$ and $q$ in $X$ there are neighborhoods $U$ and $V$ of $p$ and $q$ respectively such that $U$ and $V$ have disjoint closures.

The first example of a space having a dispersion point was given by Knaster and Kuratowski [9]. Such spaces have been investigated by Erdös [3] and Wilder [19]. Roy [15] has given an example of a countable connected Urysohn space having a dispersion point. Kirch [8] has given an example of a locally connected countable connected Hausdorff space. Stone [17] has also announced the construction of such an example. Recently Franklin and Krishnarao [5] have announced an interesting application of such an example due to F. B. Jones (unpublished).

The examples presented here are obtained by extending a metric space by adjoining countably many limit points and extending the distance function to a quasimetric. A real valued function $D(x, y)$ is a quasimetric for a topological space provided that for points $x, y, z$ of the space:

1. $D(x, y) \geq 0$, the equality holding iff $x = y$.
2. $D(x, y) + D(y, z) \geq D(x, z)$.

Presented in part to the Society, September 1, 1969 under the title A countable Urysohn space with an explosion point; received by the editors November 3, 1969.

AMS 1968 subject classifications. Primary 5401, 5440, 5455; Secondary 5423, 5435.

Key words and phrases. Countable connected Hausdorff space, dispersion point, locally connected space, Urysohn space, quasimetric, Menger-Urysohn dimension, Lebesgue dimension, completion of the rationals.

355
3. The collection of \( \varepsilon \)-balls, \( B(x, \varepsilon) = \{ y : D(x, y) < \varepsilon \} \), is a base for the topology of the space.

Therefore a symmetric quasimetric is a metric. There are just two essentially different ways of writing the triangle inequality (2); one implies symmetry — \( D((x, y) = D(y, x) \)— the other, as we have written it, does not.

The following characterization, originally due to Ribeiro [16], indicates the severity of the existence of a quasimetric. A \( T_1 \) space \( X \) is quasimetrizable iff each point \( p \) has a base for its neighborhood system \( X = G_1(p), G_2(p), \ldots \) such that for any points \( x \) and \( y \), \( x \in G_{n+1}(y) \) implies \( G_{n+1}(x) \subseteq G_n(y) \) for \( n = 1, 2, \ldots \). Relations between Moore, metric and quasimetric spaces have been investigated by Stoltenberg [10]; related work may also be found in [4] and [14].

The following examples indicate that a quasimetric space can be quite pathological even though its distance function is the Euclidean metric when restricted to a dense open subspace.

2. A countable locally connected quasimetric extension of the rationals. This connected Hausdorff space has differing inductive and covering dimensions and is not a Urysohn space.

Construction. Let \( Q \) be the set of rational points on the \( x \)-axis, and \( M \) the set of points above the \( x \)-axis with both coordinates rational. Let \( m \) be a point of \( M \). Define \( F(m) = \{ r, s \} \) where \( r \) and \( s \) are the points of the \( x \)-axis which together with \( m \) are the vertices of an equilateral triangle. Note that \( F(m) \) does not intersect \( S = Q \cup M \).

Let \( x \) and \( y \) be points of \( S \). If \( x \) is a point of \( Q \) and \( y \) a point of \( M \), define \( D(x, y) = d(x, z) + d(z, y) \) and \( D(y, x) = d(z, x) \) where \( z \) is the point of \( F(y) \) closest to \( x \) and \( d \) is the usual metric for the plane. Assume \( x \) and \( y \) are distinct points of \( M \). Note that \( F(x) \) and \( F(y) \) are disjoint. Let \( z \) and \( w \) be points of \( F(x) \) and \( F(y) \) respectively such that \( d(z, w) = d[F(x), F(y)] \). Define \( D(x, y) = d(z, w) + d(w, y) \). Finally, if \( x = y \) or \( x \) and \( y \) are points of \( Q \), define \( D(x, y) = d(x, y) \). \( D \) clearly satisfies properties (1) and (2) above. The \( \varepsilon \)-balls then form a base for some topology of \( S \). Assume \( S \) has this topology. \( S \) is clearly a Hausdorff space.

Lemma 1. Each \( \varepsilon \)-ball is connected.

Proof. Suppose to the contrary for some \( p \) in \( S \) and positive \( \varepsilon \) there are nonempty disjoint open sets \( U \) and \( V \) such that \( U \cup V = B(p, \varepsilon) \). Assume \( p \) is a point of \( Q \). Then \( B(p, \varepsilon) \) contains the interval of rationals \( (p - \varepsilon, p + \varepsilon) \cap Q \). There are open intervals \( G \) and \( H \) on the \( x \)-axis, contained in \( U \) and \( V \) respectively, such that the usual
distance from \(G\) to \(H\) is less than \(\epsilon/3\). \(B(p, \epsilon)\) contains all points of \(M\) with second coordinates less than \(\epsilon/3\) which lie in an equilateral triangle which has base \((p - \epsilon, p + \epsilon)\). Thus there is a point \(q\) in \(M \cap B(p, \epsilon)\) such that \(F(q)\) intersects both \(G\) and \(H\). Thus \(q\) is a limit point of both \(U\) and \(V\), a contradiction.

Therefore \(p\) is in \(M\). In this case, \(B(p, \epsilon)\) consists of \(p\) together with two sets each like the \(\epsilon\)-ball in the above. By a similar argument, each of the two sets is connected. Furthermore \(p\) is a limit point of the two sets. This involves a contradiction.

**Corollary.** \(S\) is a countable connected Hausdorff space which is locally connected, quasimetric and contains \(Q\) as a dense subspace.

3. A countable connected quasimetric extension of \(Q \times Q\) having a dispersion point. This space is Urysohn and also has differing dimensions.

**Construction.** Let \(X = (Q \times Q) \cup Z \cup \{\omega\}\) where \(Q\) denotes the rationals, \(Z\) the integers and \(\omega = (\pi, \sqrt{2})\). \(X\) endowed with the topology described below is the desired example. Let \(F\) be a function from \(Z\) into the plane such that (1) for each \(z\) in \(Z\), \(F(z)\) is not in \(X\) and the only image of \(F\) on the line \(\omega F(z)\) is \(F(z)\), and (2) \(F[Z]\) is dense in the plane.

For each \(z\) in \(Z\), let \(G(z)\) be the midpoint of the line segment \(\omega F(z)\). A quasimetric \(D\) for \(X\) is defined next in terms of the usual metric \(d\) for the plane. First let \(d^*\) be the usual bounded metric for the plane, \(d^* = d/1 + d\). Let \(x\) and \(y\) be points of \(X\), let \(a\) and \(b\) be points of \(X - Z\) and let \(z\) be a point of \(Z\). Define \(D\) as follows.

1. \(D(a, b) = d^*(a, b)\).
2. \(D(z, a) = \min[d^*(a, F(z)), d^*(a, G(z))]\).
3. \(D(x, y) = 1\) for \(x \neq y\).
4. \(D(x, x) = 0\).

The triangle inequality and positive definite property follow immediately. Let \(X\) have the topology induced by the quasimetric \(D\).

Notice if \(\epsilon\) is a positive number less than one, an \(\epsilon\)-ball with center a point of \(M = (Q \times Q) \cup \{\omega\}\) is just the common part of \(M\) and an open disc with center the point. An \(\epsilon\)-ball with center a point \(z\) of \(Z\) is just the union of two such rational discs with centers \(F(z)\) and \(G(z)\) together with \(z\). Thus \(Q \times Q\) is dense in \(X\), \(Z\) is closed in \(X\), and each have their usual topologies as subspaces of \(X\).

**Lemma 2.** \(X\) is connected.

**Proof.** Suppose \(X\) is the union of two nonempty disjoint open sets \(U\) and \(V\) such that \(U\) contains \(\omega\). Since each point of \(Z\) is a limit point of \(Q \times Q\), there is a point \(x_0\) in \(Q \times Q \cap V\). For each point \(x\) in the
plane, let $S(x, \epsilon)$ denote the $\epsilon$-disc with center $x$ relative to the usual metric $d$. There exist a point $z_1$ of $Z$ and a positive $\epsilon_1$ such that $S(G(z_1), \epsilon_1) \cap Q \times Q$ lies in $V$ and $\epsilon_1 < t/2$ where $t = d(\omega, F(z_1))$. Let $x_1$ be a common point of $S(G(z_1), t)$ and $Q \times Q$. There exist a point $z_2$ of $Z$ and a positive $\epsilon_2 < t/2^2$ such that $F(z_2)$ is in $S(G(z_1), \epsilon_1)$ and $S(G(z_2), \epsilon_2)$ \cap $Q \times Q$ lies in $V$. Let $x_2$ be a common point of $S(G(z_2), \epsilon_2)$ and $Q \times Q$. Continue this process. It is easily seen that for each natural number $n$, $d(\omega, x_n) < [n+1]t/2^n$. Therefore $x_1, x_2, \ldots$ converges to $\omega$ in the plane. Therefore $\omega$ is a limit point of $V$ in $X$, which is a contradiction.

**Lemma 3.** If $x$ and $y$ are points of $X - \{\omega\}$, then $X - \{\omega\}$ is the union of disjoint open sets $U$ and $V$ containing $x$ and $y$ respectively.

**Proof.** Since $\omega = (\sqrt{2}, \pi)$, and $\pi$ is transcendental, a line through $\omega$ containing a point of $Q \times Q$ cannot have rational slope. Therefore each line through $\omega$ contains at most one point of $Q \times Q$. Let $L$ be a line through $\omega$ such that $L$ does not intersect $(Q \times Q) \cup F[Z]$ and (1) $L$ separates $x$ and $y$ if both are in $Q \times Q$, (2) $L$ separates $F(x)$ and $F(y)$ if $x$ and $y$ are in $Z$, and (3) $L$ separates $F(x)$ and $y$ if $x$ is in $Z$ and $y$ is in $Q \times Q$. Let $M_i$ be the set of all points of $Q \times Q$ on one side of $L$ and $M_2$ the set of all such points on the other. Let $N_i$ be the set of all $z$ in $Z$ such that $F(z)$ is on the $M_i$ side of $L$ for $i = 1, 2$. Let $U = M_1 \cup N_1$ and $V = M_2 \cup N_2$. Then $U$ and $V$ are disjoint and open in $X - \omega$, $X - \omega = U \cup V$, and $x$ is in one of $U$ and $V$ and $y$ is in the other.

**Lemma 4.** $X$ is a Urysohn space.

**Proof.** Suppose $x$ is a point of $X - Z$. Let $U$ and $V$ be the intersections of $X - Z$ with open discs in the plane having centers $x$ and $\omega$ and each having radius $\epsilon = d(\omega, x)/4$. Clearly, no point of $X - Z$ is a limit point in $X$ of both $U$ and $V$. Suppose some point $z$ in $Z$ is a limit point of $U$. Then $d(x, F(z)) \leq \epsilon$ or $d(x, G(z)) \leq \epsilon$. In either case $d(\omega, F(z)) > \epsilon$ and $d(\omega, G(z)) > \epsilon$. Thus $z$ is not a limit point of $V$. Therefore $U \cap V = \emptyset$.

Now suppose $x$ is a point of $Z$. Let $U_1$, $U_2$ and $V$ be the intersections of $X - Z$ with open discs in the plane having centers $F(x)$, $G(x)$ and $\omega$ respectively and each having radius $d(G(x), \omega)/4$. From the first part of the proof, $U_1 \cap V = U_2 \cap V = \emptyset$. Let $U = U_1 \cup U_2 \cup \{x\}$. Then $U \cap V = (U_1 \cap V) \cup (U_2 \cap V) \cup \{x\} \cap V = \emptyset$.

Finally, suppose $x$ and $y$ are points of $X - \omega$. From Lemma 3, there are disjoint open sets $U_1$ and $V$ containing $x$ and $y$ respectively such that $X - \omega = U_1 \cup V$. There are disjoint open sets $U_2$ and $V_2$ containing $x$ and $\omega$ respectively. Let $U = U_1 \cap U_2$. Then $U \cap V = \emptyset$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Corollary. X is a countable connected quasimetric Urysohn space which has a dispersion point and contains \( Q \times Q \) as a dense subspace.

4. Dimension.

Lemma 5. S and X are each one dimensional in the Menger-Urysohn sense.

Proof. In each space, each point has arbitrarily small \( \varepsilon \)-neighborhoods with boundaries consisting of isolated points in the relative topology.

Lemma 6. S and X are each infinite dimensional in the Lebesgue sense.

Proof. Let \( P \) be the set of all points in \( S \) with second coordinate greater than 1. Then \( P \) is closed and homeomorphic to \( Z \). Let \( n \) be a positive integer. Then \( P \) can be partitioned into subsets \( P_1, \ldots, P_n \) such that for each relatively open set \( U \) in \( Q \), \( P_i \) contains a limit point of \( U \) for \( i = 1, \ldots, n \). Let \( U_i = (S - P) \cup P_i \) for \( i = 1, \ldots, n \). Then \( U_1, \ldots, U_n \) is an open cover of \( S \). Suppose \( \mathcal{G} \) is an open cover of \( S \) which refines this cover. Let \( G_1 \) be a member of \( \mathcal{G} \) which lies in \( U_1 \). Then \( Q \cap G_1 \) is a nonempty open set since \( Q \) is open and dense. Let \( p_2 \) be a limit point of \( Q \cap G_1 \) in \( P_2 \). Let \( G_2 \) be a member of \( \mathcal{G} \) which contains \( p_2 \). Then \( G_2 \) lies in \( U_2 \). \( Q \cap G_1 \cap G_2 \) is nonempty since \( p_2 \) is a limit point of \( Q \cap G_1 \). Thus \( Q \cap G_1 \cap G_2 \) has a limit point \( p_3 \) in \( P_3 \). Continuing in this manner, we have distinct members \( G_1, \ldots, G_n \) of \( \mathcal{G} \) with a common point. Thus the order of \( \mathcal{G} \) is at least \( n \). Since this holds for each positive integer, \( S \) is infinite dimensional in the Lebesgue sense.

The preceding argument when modified by replacing \( P \) by \( Z \), \( Q \) by \( Q \times Q \) and \( S \) by \( X \) establishes the desired result for \( X \).

The author wishes to thank B. J. Pearson for helpful suggestions regarding the presentation of results in this note.

References


University of Victoria, Victoria, British Columbia, Canada