AN EMBEDDING THEOREM FOR HOMEOMORPHISMS
OF THE CLOSED DISC

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Abstract. If f is an orientation preserving self-homeomorphism of the closed disc D with the property that if x, y ∈ D − N, where the set of fixed points N is finite and contained in D − int D, then there exists an arc A ⊂ D − N joining x and y such that f^n(A) tends to a fixed point as n → ± ∞, then it is shown that f can be embedded in a continuous flow on D.

1. Preliminary remarks. Let X be a topological space and let G be a topological group. The ordered triple (X, G, F) is a dynamical system if (1) F: X × G → X is continuous, (2) F(F(x, g_1), g_2) = F(x, g_1 + g_2), and (3) F(x, e) = x for every x ∈ X. If G is the additive group of real numbers, (X, G, F) is called a continuous flow. If G is the additive group of integers, then (X, G, F) is called a discrete flow. A discrete flow determines and is determined by the homeomorphism Fix, 1).

The problem of embedding a discrete flow in a continuous flow can be stated in the following way. Let f be a self-homeomorphism of a topological space X. Find a continuous flow (Y, R, F) such that (1) F(y, 1) is invariant on a subset Z of Y, (2) Z is homeomorphic to X, and (3) F(y, 1) on Z is topologically equivalent to f on X, i.e. there is a homeomorphism h: X → Z such that h^{-1}(F(h(x), 1)) = f(x).

If the space X is allowed to be enlarged in order to accommodate F, the problem is referred to as the unrestricted problem. If X = Y = Z, the problem is referred to as the restricted problem.

The unrestricted problem is easily solved [5], [7], [8] for any topological space X and any self-homeomorphism. The two solutions cited in [5] are shown in [9] to be topologically equivalent if and only if the homeomorphism has no fixed points.

The restricted problem is only partially solved. In [2] and [6] it is proved that a self-homeomorphism f of an interval can be embedded in a continuous flow if and only if it is order preserving. In [5] necessary and sufficient conditions for a self-homeomorphism of a simple closed curve to be embedded in a continuous flow are given. In [3] and [9] sufficient conditions for a self-homeomorphism of the closed disc to be embedded in a continuous flow are given. In [1],

Received by the editors February 16, 1970.

AMS 1969 subject classifications. Primary 5482; Secondary 2240, 5480.

Key words and phrases. Embedding, discrete flows, continuous flows, homeomorphisms, closed disc.
sufficient conditions for a self-homeomorphism of the plane to be embedded in a continuous flow are given.

It is easily shown that a necessary condition for a self-homeomorphism of a subset of the plane to be embedded in a continuous flow is that it be orientation preserving. Thus, throughout this paper $f$ will denote such a homeomorphism.

**2. Embedding theorem.** Let $f$ be an orientation preserving self-homeomorphism of the closed disc $D$. It is well known that $f$ must have at least one fixed point. If in addition $f$ has the property that if $x, y \in D-N$, where the set of fixed points $N$ is finite and contained in $D-\text{int } D$, then there exists an arc $A \subseteq D-N$ joining $x$ and $y$ such that $f^n(A)$ tends to a fixed point as $n \to \pm \infty$, then it is easy to show that there exist at most two fixed points.

**Theorem.** If $f$ is a homeomorphism as described above, then $f$ can be embedded in a continuous flow.

**Proof.** The case where $f$ has exactly one fixed point is proved in [9].

Suppose $f$ has two fixed points $x_1$ and $x_2$. Suppose without loss of generality $D = \{(x, y) : x^2 + y^2 \leq 1\}$ in the plane $P_1$. Suppose $x_1 = (0, 1)$ and $x_2 = (0, -1)$. Consider the set $S = P_1 - \{ (x, y) : x = 0, |y| \geq 1 \}$. This set $S$ is homeomorphic to a plane $P_2$. Suppose $g_1 : S \to P_2$ is a homeomorphism where $\{(x, y) : x^2 + y^2 = 1, x > 0\}$ is mapped onto the $y$-axis in $P_2$. If $z$ is an element of the half-plane in $P_2$ defined by the $y$-axis not containing $g_1(D)$, then define a self-homeomorphism $T_1$ of the closed half-plane by

$$T_1((x, y)) = (x, (g_1 \circ g_1^{-1}(0, y))_y).$$

Suppose $g_2 : S \to P_3$ is a homeomorphism where $\{(x, y) : x^2 + y^2 = 1, x < 0\}$ is mapped onto the $y$-axis in plane $P_3$. If $z$ is an element of the half-plane in $P_3$ defined by the $y$-axis not containing $g_2(D)$ then define a self-homeomorphism $T_2$ of the closed half-plane by $T_2((x, y)) = (x, (g_2 \circ g_2^{-1}(0, y))_y)$. Now define $T : S \to S$ by

$$T(z) = g_1^{-1}T_1g_1(z) \quad \text{if } g_1(z) \in \text{domain of } T_1,$$

$$= f(z) \quad \text{for } z \in D,$$

$$= g_2^{-1}T_2g_2(z) \quad \text{if } g_2(z) \in \text{domain of } T_2.$$

It is clear from the definition of $T$ that $T$ is an orientation preserving self-homeomorphism of $S$ with no fixed points. It is also clear that $g_1Tg_1^{-1}$ is a self-homeomorphism of $P_1$ with exactly one fundamental region, and is therefore topologically equivalent to a translation [1, p. 71]. Thus, there is a curve $L \subseteq S$ such that $T^i(L)$ separates $S$.
for every integer \( j \), \( T^n(L) \cap T^m(L) = \emptyset \) for \( n \neq m \), if \( z \in S \), \( T^k(z) \) is in the strip bounded by \( L \) and \( T(L) \) for some unique integer \( k \), and \( g_i(L) \) intersects the \( y \)-axis in \( P_{i+1} \) exactly one point for \( i = 1, 2 \).

Let \( L_1 = L \cap D \). The region bounded by \( L_1 \), \( D - \text{Int} \ D \) from \( f(L_1) \), \( f(L_1) \), and \( D - \text{Int} \ D \) from \( f(L_1) \) to \( L_1 \) is homeomorphic to a rectangle with \( f(L_1) \) and \( L_1 \) as opposite sides. Suppose \( h \) is this homeomorphism. Now if \( x \in L_1 \) and if \( 0 \leq t < 1 \) define \( F_1(x, t) = y \) where \( h(y) = (1-t)h(x) + th(f(x)) \). If \( x \in L_1 \) and \( t \) is any real number, define \( F_2(x, t) = f^n(F_1(x, s)) \), where \( n + s = t \), \( n \) is an integer, and \( 0 \leq s < 1 \). If \( x \) is in the interior of the strip bounded by \( L_1 \) and \( f(L_1) \) or if \( x \in L_1 \), and if \( t_1 \) is any real number, define \( F_3(x, t_1) = F_2(y, t_2 + t_1) \) where \( h(y) = (1-t_2)h(f(y)) + t_2h(y) \) and \( y \in L_1 \). If \( x \) is any point in \( D - \{ x_1, x_2 \} \), define \( F(x, t) = F_3(f^{-m}(x), t + m) \), where \( f^{-m}(x) \) is in the strip bounded by \( L_1 \) and \( f(L_1) \).

Now define \( \sigma : D \times \mathbb{R} \rightarrow D \) by

\[
\sigma(x, r) = \begin{cases} 
F(x, r) & \text{if } x \in D - \{ x_1, x_2 \}, \\
= x & \text{otherwise.}
\end{cases}
\]

It is not difficult to show that \( \sigma \) is a continuous flow, and that \( \sigma(x, n) = f^n(x) \) for every \( x \in D \) and every integer \( n \).

**References**


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