AN EMBEDDING THEOREM FOR HOMEOMORPHISMS OF THE CLOSED DISC

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ABSTRACT. If f is an orientation preserving self-homeomorphism of the closed disc D with the property that if $x, y \in D-N$, where the set of fixed points N is finite and contained in $D-\text{int } D$, then there exists an arc $A \subset D-N$ joining x and y such that $f^n(A)$ tends to a fixed point as $n \to \pm\infty$, then it is shown that f can be embedded in a continuous flow on D.

1. **Preliminary remarks.** Let X be a topological space and let G be a topological group. The ordered triple (X, G, F) is a *dynamical system* if (1) $F: X \times G \to X$ is continuous, (2) $F(F(x, g_1), g_2) = F(x, g_1+g_2)$, and (3) $F(x, e) = x$ for every $x \in X$. If G is the additive group of real numbers, (X, G, F) is called a *continuous flow*. If G is the additive group of integers, then (X, G, F) is called a *discrete flow*. A discrete flow determines and is determined by the homeomorphism $F(x, 1)$.

The problem of embedding a discrete flow in a continuous flow can be stated in the following way. Let f be a self-homeomorphism of a topological space X. Find a continuous flow (Y, R, F) such that (1) $F(y, 1)$ is invariant on a subset Z of Y, (2) Z is homeomorphic to X, and (3) $F(y, 1)$ on Z is topologically equivalent to f on X, i.e. there is a homeomorphism $h: X \to Z$ such that $h^{-1}(F(h(x), 1)) = f(x)$.

If the space X is allowed to be enlarged in order to accommodate F, the problem is referred to as the *unrestricted problem*. If $X = Y = Z$, the problem is referred to as the *restricted problem*.

The unrestricted problem is easily solved [5], [7], [8] for any topological space X and any self-homeomorphism. The two solutions cited in [5] are shown in [9] to be topologically equivalent if and only if the homeomorphism has no fixed points.

The restricted problem is only partially solved. In [2] and [6] it is proved that a self-homeomorphism f of an interval can be embedded in a continuous flow if and only if it is order preserving. In [5] necessary and sufficient conditions for a self-homeomorphism of a simple closed curve to be embedded in a continuous flow are given. In [3] and [9] sufficient conditions for a self-homeomorphism of the closed disc to be embedded in a continuous flow are given. In [1],

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[4], [9] sufficient conditions for a self-homeomorphism of the plane to be embedded in a continuous flow are given.

It is easily shown that a necessary condition for a self-homeomorphism of a subset of the plane to be embedded in a continuous flow is that it be orientation preserving. Thus, throughout this paper f will denote such a homeomorphism.

2. **Embedding theorem.** Let f be an orientation preserving self-homeomorphism of the closed disc D. It is well known that f must have at least one fixed point. If in addition f has the property that if $x, y \in D-N$, where the set of fixed points N is finite and contained in $D-\operatorname{int} D$, then there exists an arc $A \subset D-N$ joining x and y such that $f^n(A)$ tends to a fixed point as $n \to \pm\infty$, then it is easy to show that there exist at most two fixed points.

THEOREM. *If f is a homeomorphism as described above, then f can be embedded in a continuous flow.*

PROOF. The case where f has exactly one fixed point is proved in [9].

Suppose f has two fixed points x_1 and x_2. Suppose without loss of generality $D=\{(x, y): x^2+y^2 \leqq 1\}$ in the plane P_1. Suppose $x_1=(0, 1)$ and $x_2=(0, -1)$. Consider the set $S=P_1-\{(x, y): x=0, |y| \geqq 1\}$. This set S is homeomorphic to a plane P_2. Suppose $g_1: S \to P_2$ is a homeomorphism where $\{(x, y): x^2+y^2=1, x>0\}$ is mapped onto the y-axis in P_2. If z is an element of the half-plane in P_2 defined by the y-axis not containing $g_1(D)$, then define a self-homeomorphism T_1 of the closed half-plane by

$$T_1((x, y)) = (x, (g_1 f g_1^{-1}(0, y))_y).$$

Suppose $g_2: S \to P_3$ is a homeomorphism where $\{(x, y): x^2+y^2=1, x<0\}$ is mapped onto the y-axis in plane P_3. If z is an element of the half-plane in P_3 defined by the y-axis not containing $g_2(D)$ then define a self-homeomorphism T_2 of the closed half-plane by $T_2((x, y)) = (x, (g_2 f g_2^{-1}(0, y))_y)$. Now define $T: S \to S$ by

$$\begin{aligned} T(z) &= g_1^{-1} T_1 g_1(z) && \text{if } g_1(z) \in \text{domain of } T_1, \\ &= f(z) && \text{for } z \in D, \\ &= g_2^{-1} T_2 g_2(z) && \text{if } g_2(z) \in \text{domain of } T_2. \end{aligned}$$

It is clear from the definition of T that T is an orientation preserving self-homeomorphism of S with no fixed points. It is also clear that $g_1 T g_1^{-1}$ is a self-homeomorphism of P_2 with exactly one fundamental region, and is therefore topologically equivalent to a translation [1, p. 71]. Thus, there is a curve $L \subset S$ such that $T^j(L)$ separates S

for every integer j, $T^n(L)\cap T^m(L)=\emptyset$ for $n\neq m$, if $z\in S$, $T^k(z)$ is in the strip bounded by L and $T(L)$ for some unique integer k, and $g_i(L)$ intersects the y-axis in P_{i+1} in exactly one point for $i=1, 2$. Let $L_1=L\cap D$. The region bounded by L_1, $D-\operatorname{Int} D$ from L_1 to $f(L_1)$, $f(L_1)$, and $D-\operatorname{Int} D$ from $f(L_1)$ to L_1 is homeomorphic to a rectangle with $f(L_1)$ and L_1 as opposite sides. Suppose h is this homeomorphism. Now if $x\in L_1$ and if $0\leqq t<1$ define $F_1(x, t)=y$ where $h(y)=(1-t)h(x)+th(f(x))$. If $x\in L_1$ and t is any real number, define $F_2(x, t)=f^n(F_1(x, s))$, where $n+s=t$, n is an integer, and $0\leqq s<1$. If x is in the interior of the strip bounded by L_1 and $f(L_1)$ or if $x\in L_1$, and if t_1 is any real number, define $F_3(x, t_1)=F_2(y, t_2+t_1)$ where $h(x)=(1-t_2)h(f(y))+t_2(h(y))$ and $y\in L_1$. If x is any point in $D-\{x_1, x_2\}$, define $F(x, t)=F_3(f^{-m}(x), t+m)$, where $f^{-m}(x)$ is in the strip bounded by L_1 and $f(L_1)$.

Now define $\sigma: D\times R\to D$ by

$$\begin{aligned}\sigma(x, r) &= F(x, r) \quad \text{if } x\in D-\{x_1, x_2\},\\ &= x \qquad\qquad \text{otherwise.}\end{aligned}$$

It is not difficult to show that σ is a continuous flow, and that $\sigma(x, n)=f^n(x)$ for every $x\in D$ and every integer n.

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