ANOTHER ZERO-FREE REGION FOR $\zeta^{(k)}(s)$[1]

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ABSTRACT. A new zero-free region in the left half-plane is found for the kth derivative of the Riemann zeta function.

It was shown in [1] that $\zeta^{(k)}(s)$, the kth derivative of the Riemann zeta function, has certain zero-free regions. Write $s=\sigma+it$. The usual theory, for any Dirichlet series, was used for the obvious region $\sigma \geqq \sigma_k$. It was also shown that for each $k \geqq 1$ and each $\epsilon > 0$, there is an $r_k = r_k(\epsilon)$ such that $\zeta^{(k)}(s) \neq 0$ for $|s| > r_k$, $\sigma < -\epsilon$ and $|t| > \epsilon$. In this paper, using the results of [1], we show the following:

THEOREM. *For $k \geqq 0$ there is an α_k so that $\zeta^{(k)}(s)$ has only real zeros for $\sigma \leqq \alpha_k$, and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1-2n \leqq \alpha_k$.*

This theorem is used in Berndt [2]. The result is well known for $k=0$. To prove the theorem, we write, as in [1], for $k \geqq 1$,

$$(-1)^k \zeta^{(k)}(1-s) = 2(2\pi)^{-s} \sum_{j=0}^{k} \Gamma^{(j)}(s) R_{jk}(s),$$

where

$$R_{jk}(s) = P_{jk}(s)\cos(\pi s/2) + Q_{jk}(s)\sin(\pi s/2),$$

$$P_{jk}(s) = \sum_{n=0}^{k} a_{jkn}\zeta^{(n)}(s),$$

$$Q_{jk}(s) = \sum_{n=0}^{k} b_{jkn}\zeta^{(n)}(s),$$

and

$$P_{kk}(s) = \zeta(s), \quad Q_{kk}(s) = 0.$$

Write

$$(-1)^k \zeta^{(k)}(1-s)/2(2\pi)^{-s} = f(s) + g(s)$$

where $f(s) = \zeta(s)\Gamma^{(k)}(s)\cos(\pi s/2)$, $g(s) = \sum_{j=0}^{k-1} \Gamma^{(j)}(s) R_{jk}(s)$. Now we

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apply Rouche's theorem to the square with vertices $2n\pm i$, $2n+2\pm i$, and $f(s)+g(s)$ will have exactly the same number of zeros inside this square as $f(s)$, provided $|f(s)|>|g(s)|$ on the boundary. Now, from [1],

$$\Gamma^{(j)}(s) = \Gamma(s)\left[\log^j s + \sum_{n=0}^{j-1} E_{nj}(s)\log^n s\right],$$

where each $E_{nj}(s)$ is $O(1/s)$, so for σ sufficiently large, $\Gamma^{(k)}(s)\neq 0$, and $f(s)$ has only the single zero in the square due to $\cos(\pi s/2)$.

Now we proceed exactly as in [1]. Setting

$$R^*_{jk}(s) = P_{jk}(s) + Q_{jk}(s)\tan(\pi s/2),$$

dividing $|f(s)|>|g(s)|$ by $|\Gamma(s)(\cos(\pi s/2))\zeta(s)\log^{k-1}s|$, and applying the triangle inequality several times, we will have $|f(s)|>|g(s)|$ on the boundary provided

$$|\log s| > \sum_{n=0}^{k-1}|E_{nk}(s)\log^{n+1-k}s| + \left|\sum_{j=0}^{k-1}{}^{*}\frac{R_{jk}(s)}{\zeta(s)}\left\{\frac{1}{\log^{k-1-j}s}+\sum_{n=0}^{j-1}\frac{E_{nj}(s)}{\log^{k-1-n}s}\right\}\right|,$$

As in [1], this inequality will hold on the upper and lower edges of the square, and also on the sides provided $|\tan(\pi s/2)|$ is bounded. But on $\sigma=2n$, or $\sigma=2n+2$,

$$|\tan(\pi s/2)| = |(e^{-\pi t}-1)/(e^{-\pi t}+1)|,$$

which is clearly bounded. Thus, $\zeta^{(k)}(s)$ has exactly one zero inside the square, and since $\zeta^{(k)}(s)$ is real on the real axis, this zero must lie on the real axis, as nonreal zeros occur in conjugate pairs, and the proof is complete.

References

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