ON ELEMENTARY GROUPS

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Abstract. Bechtell has defined a group $G$ to be elementary if the Frattini subgroup of each subgroup of $G$ is the identity. In this note we prove the following: If the derived group of $G$ is nilpotent, then necessary and sufficient conditions that $G$ be elementary are that the Frattini subgroup of some Carter subgroup $K$ of $G$ be equal to the derived group of $K$.

Bechtell in [1] has defined a group $G$ to be elementary if the Frattini subgroup of each subgroup $H$ of $G$, Fr$(H)$, is the identity. In this note we find that if $G^1$ is nilpotent then necessary and sufficient conditions that $G$ be elementary are that Fr$(G)$ = 1 and Fr$(K)$ = $K^1$ for some (hence all) Carter subgroup $K$ of $G$. Only finite solvable groups are considered here and the notation is as in [4].

Lemma 1. Let $A$ be an invariant subgroup of $G$ and $B$ be a subgroup of $G$ such that $A \subseteq B$ and $B/A$ is a Carter subgroup of $G/A$. Let $H$ be a Carter subgroup of $B$. Then $H$ is a Carter subgroup of $G$.

Proof. $H$ is nilpotent. Let $x \in N_G(H)$. Then $xA \in N_{G/A}(HA/A)$. By a remark in [2], HA/A is a Carter subgroup of B/A and B/A is nilpotent. Hence HA = B and $xA \in N_{G/A}(B/A)$. Therefore $x \in B$ and hence $x \in N_B(H) = H$.

Lemma 2. Let $G^1$ be complemented in $G$ by a subgroup $K$ such that $G^1$ and $K$ are abelian and $G^1 \cap Z(G) = 1$. Then Carter subgroups of $G$ are precisely those subgroups of $G$ which are complements to $G^1$.

Proof. $K$ is nilpotent. If $N_G(K)$ contains $K$ properly then there exists $x \in N_G(K) \cap G^1$, $x \neq 1$. Then for all $k \in K$, $xkx^{-1} \in K$ which yields $xkx^{-1}k^{-1} \in K \cap G^1 = 1$. Therefore $x \in C_G(K)$ and, since $G^1$ is abelian, $x \in Z(G)$, whence $x = 1$, a contradiction. Therefore $N_G(K) = K$ and $K$ is a Carter subgroup of $G$. If $J$ is another Carter subgroup of $G$, then $K$ and $J$ are conjugate, hence $J$ also complements $G^1$.

Corollary. Let $G^1$ be nilpotent and Fr$(G)$ = 1. Then Carter subgroups of $G$ are precisely those subgroups of $G$ which complement $G^1$.

Proof. By Satz 12 in [3], Fit$(G) = \text{Soc}(G)$, hence $G^1$ is abelian. By

Received by the editors February 5, 1970.

AMS 1969 subject classifications. Primary 2040, 2025.

Key words and phrases. Elementary group, Carter subgroup, Frattini subgroup.
Satz 7 in [3], there exists a complement $K$ to $G^1$ in $G$. Since $[K, K] \subseteq K \cap G^1 = 1$, $K$ is abelian. Furthermore $Z(G) \cap G^1 \subseteq \text{Fr}(G) = 1$. Hence $G$ satisfies the hypothesis of Lemma 2.

Lemma 3. Let $G^1$ be nilpotent. Then the following are equivalent:

1. $\text{Fr}(G) = 1$.
2. $\text{Fit}(G) = \text{Soc}(G)$ and $\text{Fit}(G)$ is complemented by a subgroup and $\text{Fr}(G) \subseteq G^1$.
3. $G^1$ is abelian, is completely reducible under inner automorphisms of $G$, is complemented by a subgroup and $\text{Fr}(G) \subseteq G^1$.

Proof. That (1) implies (2) is well known even if $G^1$ is not nilpotent.

Assume (2) holds and proceed by induction on the order of $G$. Since $G^1 \subseteq \text{Fit}(G) = \text{Soc}(G)$, $G^1$ is abelian. If every minimal invariant subgroup of $G$ is contained in $G^1$, then $G^1 = \text{Soc}(G)$ and (3) follows. Therefore let $A$ be a minimal invariant subgroup of $G$ such that $A \not\subseteq G^1$. Hence $A \not\subseteq \text{Fr}(G)$ and $A$ is complemented by a maximal subgroup, say $K$. Since $[G, A] \subseteq G^1 \cap A = 1$, $A$ is central in $G$, hence $G$ is the direct product of $A$ and $K$. $K$ inherits the conditions (2), hence $K$ satisfies (3) by induction. It now follows that $G$ also satisfies (3).

Assume (3) holds. Then $G^1$ is the direct product of minimal invariant subgroups of $G$ which we denote by $A_1, \ldots, A_s$ and $G$ is the semidirect product of $G^1$ and a subgroup, say $K$. Now $K = KA_1 \cdots A_i \cdots A_s$ is a maximal subgroup of $G$ since if $M$ is a maximal subgroup of $G$ properly containing $K_i$, then $M \cap A_i \neq 1$ and $M \cap A_i$ is invariant in $G$, hence equals $A_i$. This implies that $M = G$, a contradiction. Therefore $\text{Fr}(G) \subseteq K$ and $\text{Fr}(G) \subseteq K \cap G^1 = 1$. Hence (1) holds.

Lemma 4. Let $G^1 \subseteq \text{Soc}(G)$, $A$ be an invariant subgroup of $G$ contained in $G^1$ and $Z(G) \cap G^1 = 1$. Then $Z(G/A) \cap G^1/A$ is the identity of $G/A$.

Proof. $G^1$ is the direct product of $A$ and an invariant subgroup of $G$, say $B$. Let $xA \in Z(G/A) \cap G^1/A$. Since $xA \in Z(G/A)$, $xg^{-1}a^{-1}g \in A$ for all $g \in G$. Since $x \in G^1/A$, $x = ab$ where $a \in A$ and $b \in B$. Then $a^{-1}g^{-1}ba^{-1}g^{-1} = abgb^{-1}a^{-1}g^{-1} = a = c$ where $c = ga^{-1}g^{-1} \in A$. Hence $bgb^{-1}a^{-1}g^{-1} \in A \cap B = 1$ for all $g \in G$ and therefore $b \in Z(G) \cap G^1 = 1$. Consequently $x \in A$ and $Z(G/A) \cap G^1/A$ is the identity of $A$.

Theorem. Let $G$ be a group such that $G^1$ is nilpotent. Then $G$ is elementary if and only if $\text{Fr}(G) = 1$ and $\text{Fr}(K) = K^1$ for some Carter subgroup $K$ of $G$. 

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Proof. If $G$ is elementary, then $\text{Fr}(G) = 1$. Let $K$ be a Carter subgroup of $G$. Then $K$ complements $G^1$, hence $K$ is abelian. Therefore $K^1 = 1 = \text{Fr}(K)$.

Conversely, in order to show a contradiction, let $G$ be a group of minimal order such that $G^1$ is nilpotent, $\text{Fr}(G) = 1$, $\text{Fr}(K) = K^1$ for some Carter subgroup $K$ of $G$ and $G$ is not elementary. Then $\text{Fr}(J) = J^1$ for every Carter subgroup $J$ of $G$. Let $M$ be a subgroup of $G$. If $M \supseteq G^1$, then $M$ is invariant in $G$ and $\text{Fr}(M) \subseteq \text{Fr}(G) = 1$. Assume that $G^1 \not\subseteq M$ and that $M G^1 \subseteq G$. Since $G^1 \subseteq M G^1$, $\text{Fr}(M G^1) = 1$ and $G / M G^1$ is abelian, hence $G / M G^1$ is its own Carter subgroup. Then, if $H$ is a Carter subgroup of $M G^1$, $H$ is also a Carter subgroup of $G$ by Lemma 1, hence by assumption $\text{Fr}(H) = H^1$. By the minimality of $G$, $M G^1$ is elementary, hence $\text{Fr}(M) = 1$.

Suppose now that $M G^1 = G$. Since $G^1$ is abelian, $G^1 \cap M$ is an invariant subgroup in $G$ and $M / (G^1 \cap M)$ complements $G^1 / (G^1 \cap M) = (G^1 / (G^1 \cap M))^1$ in $G / (G^1 \cap M)$. Since $\text{Fr}(G) = 1$, $G^1 \cap Z(G) = 1$ and, by Lemma 3, $G^1 \subseteq \text{Soc}(G)$. Therefore, by Lemma 4, $(G / (G^1 \cap M))^1 \cap Z(\text{Fr}(G) / (G^1 \cap M))$ is the identity of $G / (G^1 \cap M)$. Hence $M / (G^1 \cap M)$ is a Carter subgroup of $G / (G^1 \cap M)$ by Lemma 2. Let $K$ be a Carter subgroup of $M$. By Lemma 1, $K$ is a Carter subgroup of $G$. Hence $K$ is a complement to $G^1$ and $K (G^1 \cap M) = M$ since $K \subseteq M$. Therefore, $K$ is a complement to $G^1 \cap M$ in $M$. $G^1 \cap M$ is completely reducible under the inner automorphisms of $M$ and $M^1 \subseteq G^1 \cap M$, hence $M^1$ is completely reducible under the inner automorphisms of $M$. Let $J$ be a complement to $M^1$ in $G^1 \cap M$ which is invariant under the inner automorphisms of $M$. Then $J \subseteq Z(M) \cap G^1$, hence $G^1 \cap M = M^1 J \subseteq M^1 (G^1 \cap Z(M)) \subseteq G^1 \cap M$. Therefore $G^1 \cap M = M^1 (G^1 \cap Z(M))$. But $Z(M) \subseteq N_M(K) = K$, hence $G^1 \cap Z(M) \subseteq G^1 \cap K = 1$ and $G^1 \cap M = M^1$. Furthermore, $\text{Fr}(K) = K^1$ as a Carter subgroup of $G$. Now $M$ satisfies part (3) of Lemma 3, hence $\text{Fr}(M) = 1$. Consequently, $\text{Fr}(M) = 1$ for every subgroup $M$ of $G$ and $G$ is elementary, a contradiction. This completes the proof.

References


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