# ON ELEMENTARY GROUPS

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ABSTRACT. Bechtell has defined a group $G$ to be elementary if the Frattini subgroup of each subgroup of $G$ is the identity. In this note we prove the following: If the derived group of $G$ is nilpotent, then necessary and sufficient conditions that $G$ be elementary are that the Frattini subgroup of $G$ be the identity and that the Frattini subgroup of some Carter subgroup $K$ of $G$ be equal to the derived group of $K$.

Bechtell in [1] has defined a group $G$ to be elementary if the Frattini subgroup of each subgroup $H$ of $G$, $\mathrm{Fr}(H)$, is the identity. In this note we find that if $G^1$ is nilpotent then necessary and sufficient conditions that $G$ be elementary are that $\mathrm{Fr}(G)=1$ and $\mathrm{Fr}(K)=K^1$ for some (hence all) Carter subgroup $K$ of $G$. Only finite solvable groups are considered here and the notation is as in [4].

LEMMA 1. *Let $A$ be an invariant subgroup of $G$ and $B$ be a subgroup of $G$ such that $A\subseteq B$ and $B/A$ is a Carter subgroup of $G/A$. Let $H$ be a Carter subgroup of $B$. Then $H$ is a Carter subgroup of $G$.*

PROOF. $H$ is nilpotent. Let $x\in N_G(H)$. Then $xA\in N_{G/A}(HA/A)$. By a remark in [2], $HA/A$ is a Carter subgroup of $B/A$ and $B/A$ is nilpotent. Hence $HA=B$ and $xA\in N_{G/A}(B/A)$. Therefore $x\in B$ and hence $x\in N_B(H)=H$.

LEMMA 2. *Let $G^1$ be complemented in $G$ by a subgroup $K$ such that $G^1$ and $K$ are abelian and $G^1\cap Z(G)=1$. Then Carter subgroups of $G$ are precisely those subgroups of $G$ which are complements to $G^1$.*

PROOF. $K$ is nilpotent. If $N_G(K)$ contains $K$ properly then there exists $x\in N_G(K)\cap G^1$, $x\neq 1$. Then for all $k\in K$, $xkx^{-1}\in K$ which yields $xkx^{-1}k^{-1}\in K\cap G^1=1$. Therefore $x\in C_G(K)$ and, since $G^1$ is abelian, $x\in Z(G)$, whence $x=1$, a contradiction. Therefore $N_G(K)=K$ and $K$ is a Carter subgroup of $G$. If $J$ is another Carter subgroup of $G$, then $K$ and $J$ are conjugate, hence $J$ also complements $G^1$.

COROLLARY. *Let $G^1$ be nilpotent and $\mathrm{Fr}(G)=1$. Then Carter subgroups of $G$ are precisely those subgroups of $G$ which complement $G^1$.*

PROOF. By Satz 12 in [3], $\mathrm{Fit}(G)=\mathrm{Soc}(G)$, hence $G^1$ is abelian. By

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Satz 7 in [3], there exists a complement $K$ to $G^1$ in $G$. Since $[K, K] \subseteq K \cap G^1 = 1$, $K$ is abelian. Furthermore $Z(G) \cap G^1 \subseteq \mathrm{Fr}(G) = 1$. Hence $G$ satisfies the hypothesis of Lemma 2.

LEMMA 3. *Let $G^1$ be nilpotent. Then the following are equivalent:*

(1) $\mathrm{Fr}(G) = 1$.

(2) $\mathrm{Fit}(G) = \mathrm{Soc}(G)$ *and* $\mathrm{Fit}(G)$ *is complemented by a subgroup and* $\mathrm{Fr}(G) \subseteq G^1$.

(3) $G^1$ *is abelian, is completely reducible under inner automorphisms of $G$, is complemented by a subgroup and* $\mathrm{Fr}(G) \subseteq G^1$.

PROOF. That (1) implies (2) is well known even if $G^1$ is not nilpotent.

Assume (2) holds and proceed by induction on the order of $G$. Since $G^1 \subseteq \mathrm{Fit}(G) = \mathrm{Soc}(G)$, $G^1$ is abelian. If every minimal invariant subgroup of $G$ is contained in $G^1$, then $G^1 = \mathrm{Soc}(G)$ and (3) follows. Therefore let $A$ be a minimal invariant subgroup of $G$ such that $A \not\subseteq G^1$. Hence $A \not\subseteq \mathrm{Fr}(G)$ and $A$ is complemented by a maximal subgroup, say $K$. Since $[G, A] \subseteq G^1 \cap A = 1$, $A$ is central in $G$, hence $G$ is the direct product of $A$ and $K$. $K$ inherits the conditions (2), hence $K$ satisfies (3) by induction. It now follows that $G$ also satisfies (3).

Assume (3) holds. Then $G^1$ is the direct product of minimal invariant subgroups of $G$ which we denote by $A_1, \cdots, A_s$ and $G$ is the semidirect product of $G^1$ and a subgroup, say $K$. Now $K_i = KA_1 \cdots \hat{A}_i \cdots A_s$ is a maximal subgroup of $G$ since if $M$ is a maximal subgroup of $G$ properly containing $K_i$, then $M \cap A_i \neq 1$ and $M \cap A_i$ is invariant in $G$, hence equals $A_i$. This implies that $M = G$, a contradiction. Therefore $\mathrm{Fr}(G) \subseteq K$ and $\mathrm{Fr}(G) \subseteq K \cap G^1 = 1$. Hence (1) holds.

LEMMA 4. *Let $G^1 \subseteq \mathrm{Soc}(G)$, $A$ be an invariant subgroup of $G$ contained in $G^1$ and $Z(G) \cap G^1 = 1$. Then $Z(G/A) \cap G^1/A$ is the identity of $G/A$.*

PROOF. $G^1$ is the direct product of $A$ and an invariant subgroup of $G$, say $B$. Let $xA \in Z(G/A) \cap G^1/A$. Since $xA \in Z(G/A)$, $xgx^{-1}g^{-1} \in A$ for all $g \in G$. Since $xA \in G^1/A$, $x = ab$ where $a \in A$ and $b \in B$. Then $xgx^{-1}g^{-1} = abgb^{-1}a^{-1}g^{-1} = acbgb^{-1}g^{-1}$ where $c = ga^{-1}g^{-1} \in A$. Hence $bgb^{-1}g^{-1} \in A \cap B = 1$ for all $g \in G$ and therefore $b \in Z(G) \cap G^1 = 1$. Consequently $x \in A$ and $Z(G/A) \cap G^1/A$ is the identity of $A$.

THEOREM. *Let $G$ be a group such that $G^1$ is nilpotent. Then $G$ is elementary if and only if* $\mathrm{Fr}(G) = 1$ *and* $\mathrm{Fr}(K) = K^1$ *for some Carter subgroup $K$ of $G$.*

PROOF. If $G$ is elementary, then $\mathrm{Fr}(G)=1$. Let $K$ be a Carter subgroup of $G$. Then $K$ complements $G^1$, hence $K$ is abelian. Therefore $K^1=1=\mathrm{Fr}(K)$.

Conversely, in order to show a contradiction, let $G$ be a group of minimal order such that $G^1$ is nilpotent, $\mathrm{Fr}(G)=1$, $\mathrm{Fr}(K)=K^1$ for some Carter subgroup $K$ of $G$ and $G$ is not elementary. Then $\mathrm{Fr}(J)=J^1$ for every Carter subgroup $J$ of $G$. Let $M$ be a subgroup of $G$. If $M\supseteq G^1$, then $M$ is invariant in $G$ and $\mathrm{Fr}(M)\subseteq\mathrm{Fr}(G)=1$. Assume that $G^1\not\subseteq M$ and that $MG^1\subset G$. Since $G^1\subseteq MG^1$, $\mathrm{Fr}(MG^1)=1$ and $G/MG^1$ is abelian, hence $G/MG^1$ is its own Carter subgroup. Then, if $H$ is a Carter subgroup of $MG^1$, $H$ is also a Carter subgroup of $G$ by Lemma 1, hence by assumption $\mathrm{Fr}(H)=H^1$. By the minimality of $G$, $MG^1$ is elementary, hence $\mathrm{Fr}(M)=1$.

Suppose now that $MG^1=G$. Since $G^1$ is abelian, $G^1\cap M$ is an invariant subgroup in $G$ and $M/(G^1\cap M)$ complements $G^1/(G^1\cap M)=(G/(G^1\cap M))^1$ in $G/(G^1\cap M)$. Since $\mathrm{Fr}(G)=1$, $G^1\cap Z(G)=1$ and, by Lemma 3, $G^1\subseteq\mathrm{Soc}(G)$. Therefore, by Lemma 4, $(G/(G^1\cap M))^1\cap Z(G/(G^1\cap M))$ is the identity of $G/(G^1\cap M)$. Hence $M/(G^1\cap M)$ is a Carter subgroup of $G/(G^1\cap M)$ by Lemma 2. Let $K$ be a Carter subgroup of $M$. By Lemma 1, $K$ is a Carter subgroup of $G$. Hence $K$ is a complement to $G^1$ and $K(G^1\cap M)=M$ since $K\subseteq M$. Therefore, $K$ is a complement to $G^1\cap M$ in $M$. $G^1\cap M$ is completely reducible under the inner automorphisms of $M$ and $M^1\subseteq G^1\cap M$, hence $M^1$ is completely reducible under the inner automorphisms of $M$. Let $J$ be a complement to $M^1$ in $G^1\cap M$ which is invariant under the inner automorphisms of $M$. Then $J\subseteq Z(M)\cap G^1$, hence $G^1\cap M=M^1J\subseteq M^1(G^1\cap Z(M))\subseteq G^1\cap M$. Therefore $G^1\cap M=M^1(G^1\cap Z(M))$. But $Z(M)\subseteq N_M(K)=K$, hence $G^1\cap Z(M)\subseteq G^1\cap K=1$ and $G^1\cap M=M^1$. Furthermore, $\mathrm{Fr}(K)=K^1$ as a Carter subgroup of $G$. Now $M$ satisfies part (3) of Lemma 3, hence $\mathrm{Fr}(M)=1$. Consequently, $\mathrm{Fr}(M)=1$ for every subgroup $M$ of $G$ and $G$ is elementary, a contradiction. This completes the proof.

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