OPERATORS FROM BANACH SPACES TO COMPLEX INTERPOLATION SPACES

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ABSTRACT. Given a closed linear operator A with dense domain in a Banach space X, M. Schechter [4] utilized the Lebesgue integral to construct a family of bounded linear operators from X to the Calderón complex interpolation space $(X, D(A))_s$ [2], where $D(A)$, the domain of A in X, is a Banach space under the norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

In this paper we utilize the complex functional calculus, which provides a more natural setting, to construct a similar family of operators. At the same time we achieve a strengthening of the Schechter result, for in the proof of our theorem we make no use of the adjoint A^* of A and consequently do not require the domain of A to be dense in X. A completely analogous procedure would permit the removal from the Schechter theorem, referred to above, of the hypothesis that the domain of A is dense in X.

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In this paper we utilize the complex functional calculus, which provides a more natural setting, to construct a similar family of operators. At the same time we achieve a strengthening of the Schechter result, for in the proof of our theorem we make no use of the adjoint A^* of A and consequently do not require the domain of A to be dense in X. A completely analogous procedure would permit the removal from the Schechter theorem referred to above of the hypothesis that the domain of A is dense in X.

THEOREM. (a) *Let X be a complex Banach space. Let A be a closed linear operator in X such that the resolvent set of A is nonempty, and the spectrum of A is contained in a Cauchy domain; that is, a subset D of the complex plane with the following properties:*

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(1) *D is open.*

(2) *D has a finite number of components, the closures of any two of which are disjoint.*

(3) *The boundary $B(D)$ of D is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect.*

(b) *Let $\phi(w, z)$ be a complex-valued function defined for each w in the closure of the open strip S in the complex plane between* 0 *and* 1, *and for each z in a set N containing a neighborhood of the closure of the Cauchy domain D, together with a neighborhood of infinity. Suppose further that $\phi(w, z)$ satisfies the following:*

(1) *There exists a constant C such that for w_1, w_2 in the closure of the open strip S, and z in $B(D)$, then*

$$|\phi(w_1, z) - \phi(w_2, z)| \leqq C|w_1 - w_2|.$$

(2) *Let w in the closure of S be fixed, then $\phi(w, z)$ is an analytic function of z in a neighborhood of the closure of D and at infinity.*

(3) *For each w in S, and each z in $B(D)$, the partial derivative $\phi_w(w, z)$ exists and is continuous in z on $B(D)$. We assume further that*

$$|(\phi(w + h, z) - \phi(w, z))/h - \phi_w(w, z)|$$

converges to 0 *uniformly in z on $B(D)$ as h approaches* 0.

(4) *For $j=0$, 1, there exist constants K_j such that for t real and z in $B(D)$,*

$$|\phi(j + it, z)| \leqq K_j.$$

(c) *$B(D)$ is rectifiable by assumption* (a). *Let L denote the length of $B(D)$. Let M denote the maximum value achieved by the continuous function $\|(z-A)^{-1}\|$ on the compact set $B(D)$. Let s satisfying $0<s<1$ be fixed. $B(D)$ is compact. Let T denote the maximum of $\{|z| \mid z$ is a member of $B(D)\}$.*

Then $\phi(w, z)$ induces a bounded linear mapping $\Phi_s(A)$ from X to the Calderón complex interpolation space $(X, D(A))_s$ with norm $\leqq \max\{K_0ML, K_1L(1+M+TM)\}$.

Remark. Observe that in particular an arbitrary constant function $\phi(w, z)$ satisfies the four conditions in assumption (b). We note further that $\phi(w, z)$ satisfies all conditions in assumption (b) if $\phi(w, z) = f(w)g(z)$ where $f(w)$ is such that:

(1) $f(w)$ is analytic in the open strip S.

(2) $f(w)$ satisfies a Lipschitz condition in the closure of S.

(3) There exist constants K_j, $j=0$, 1, such that $|f(j+it)| \leqq K_j$ for all real t.

(For example $f(w)=1/(w+1)^n$, n a positive integer, and $g(z)$ is analytic in a neighborhood of the closure of D, and at infinity.) (For example $g(z)=1/(z-a)^n$, where n is a positive integer and "a" is any point of the resolvent set of A which does not belong to the closure of the Cauchy domain D.)

PROOF OF THEOREM. We first recall the definition of the Calderón interpolation space $(X, D(A))_s$ [1].

(1) $X+D(A)=\{x_0+x_1 \mid x_0 \in X \text{ and } x_1 \in D(A)\}$ is a Banach space under the norm

$$\|x\|_{X+D(A)} = \inf_{x_0+x_1=x} \{\|x_0\|_X + \|x_1\|_{D(A)}\}.$$

(2) Let $H(X, D(A))$ be the set of all functions f from the closure of the open strip S to $X+D(A)$ which satisfy:

(i) f is analytic in S.

(ii) f is continuous on the closure of S.

(iii) For each real t, $f(it) \in X$, $f(1+it) \in D(A)$ and there exist constants C_0, C_1 such that for all real t $\|f(it)\|_X \leq C_0$ and $\|f(1+it)\|_{D(A)} \leq C_1$.

$H(X, D(A))$ is a Banach space under the norm

$$\|f\|_{H(X,D(A))} = \max\left\{\sup_t \|f(it)\|_X, \sup_t \|f(1+it)\|_{D(A)}\right\}.$$

(3) $(X, D(A))_s = \{x \mid x=f(s), f \in H(X, D(A))\}$ is a Banach space under the norm $\|x\|_{(X,D(A))_s} = \inf_{f(s)=x}\{\|f\|_{H(X,D(A))}\}$.

We show in the sequel that each of the following is valid:

(I) For w fixed in the closure of S,

$$\phi(w, A) = \int_{B(D)} \phi(w, z)(z-A)^{-1}dz$$

is a bounded linear operator mapping X into itself.

(II) For x fixed in X, $\phi(w, A)x$ is a continuous mapping of the closure of S into X.

(III) For x fixed in X, $\phi(w, A)x$ is an analytic mapping of S into X.

(IV) For x fixed in X, $\phi(w, A)x$ is a function in $H(X, D(A))$ satisfying

$$\|\phi(w, A)x\|_{H(X,D(A))} \leq \max\{K_0ML, K_1L(1+M+TM)\}\|x\|.$$

Assuming for the time being I, II, III and IV above, it is clear that mapping $\Phi_s(A)$ of X into $(X, D(A))_s$ defined by

$$\Phi_s(A)x = [\phi(w, A)x](s) = \phi(s, A)x$$

is the required bounded operator of our theorem since

(1) For x_1, x_2 in X and a_1, a_2 scalars,

$$\begin{aligned}\Phi_s(A)(a_1x_1 + a_2x_2) &= [\phi(w, A)(a_1x_1 + a_2x_2)](s)\\ &= \phi(s, A)(a_1x_1 + a_2x_2)\\ &= a_1\phi(s, A)x_1 + a_2\phi(s, A)x_2\\ &= a_1\Phi_s(A)x_1 + a_2\Phi_s(A)x_2.\end{aligned}$$

Thus, the operator $\Phi_s(A)$ is linear.

(2) For x in X,

$$\begin{aligned}\|\Phi_s(A)x\|_{(X,D(A))_s} &= \inf_{f(s)=\Phi_s(A)x}\{\|f\|_{H(X,D(A))}\}\\ &\leqq \|\phi(w, A)x\|_{H(X,D(A))}\\ &\leqq \max\{K_0ML, K_1L(1 + M + TM)\}\|x\|.\end{aligned}$$

Hence, $\Phi_s(A)$ is bounded with norm satisfying

$$\|\Phi_s(A)\| \leqq \max\{K_0ML, K_1L(1 + M + TM)\}.$$

We proceed now to establish the validity of I, II, III and IV.

(1) Statement (I) is a direct consequence of assumptions (a), (b), part (2), and a classical result of complex functional analysis [**3**].

(2) To verify (II), we recall that $B(D)$, the boundary of D, is rectifiable with length L and that $\|(z-A)^{-1}\|$ is continuous on $B(D)$ with maximum value M. Let x in X be fixed. Let w, w_0 be points of the closure of the open strip S. Then

$$\begin{aligned}&\|\phi(w, A)x - \phi(w_0, A)x\|\\ &= \left\|\left(\int_{B(D)} \phi(w, z)(z - A)^{-1}dz\right)x - \left(\int_{B(D)} \phi(w_0, z)(z - A)^{-1}dz\right)x\right\|\\ &= \left\|\left(\int_{B(D)} (\phi(w, z) - \phi(w_0, z))(z - A)^{-1}dz\right)x\right\|\\ &\leqq \left\{\int_{B(D)} |\phi(w, z) - \phi(w_0, z)|\, \|(z - A)^{-1}\|\, |dz|\right\}\|x\|\\ &\leqq C|w - w_0|\left(\int_{B(D)} \|(z - A)^{-1}\|\, |dz|\right)\|x\|\end{aligned}$$

(by the Lipschitz condition of assumption (b))

$$\leqq CML|w - w_0|\,\|x\|.$$

The continuity of $\phi(w, A)x$ on the closure of S is now evident.

(3) We establish (III) as follows. In accordance with assumption (b), part (3), the partial derivative $\phi_w(w, z)$ exists for w in S and z in $B(D)$, and is continuous in z on $B(D)$ for each fixed w in S. Therefore, $\int_{B(D)}\phi_w(w, z)(z-A)^{-1}dz$ exists.

Now let a positive number ϵ be given. Then for complex h sufficiently small in absolute value one has

$$\left\|(\phi(w+h, A)x-\phi(w, A)x)/h-\left(\int_{B(D)}\phi_w(w, z)(z-A)^{-1}dz\right)x\right\|$$

$$=\left\|\left\{\int_{B(D)}\{(\phi(w+h, z)-\phi(w, z))/h-\phi_w(w, z)\}(z-A)^{-1}dz\right\}x\right\|$$

$$\leqq\left\{\int_{B(D)}|(\phi(w+h, z)-\phi(w, z))/h-\phi_w(w, z)|\,\|(z-A)^{-1}\|\,|dz|\right\}\|x\|$$

$$\leqq\left\{\int_{B(D)}\epsilon\|(z-A)^{-1}\|\,|dz|\right\}\|x\| \quad \text{(by assumption (b), part (3))}$$

$$\leqq \epsilon ML\|x\|.$$

The analyticity of $\phi(w, A)x$ in S is now clear.

(4) Given the previous results concerning the continuity and analyticity of $\phi(w, A)x$, one is able to show that $\phi(w, A)x$ is a member of $H(X, D(A))$ by ascertaining the following:

(i) For t real, $\phi(1+it, A)x$ is a member of $D(A)$.

(ii) There exist constants M_0, M_1 such that

$$\|\phi(1+it, A)x\|_{D(A)} \leqq M_1 \quad \text{for all real } t, \quad \text{and}$$
$$\|\phi(it, A)x\| \leqq M_0 \quad \text{for all real } t.$$

Let t be a fixed real number and let x be a fixed member of X. Let $y=\phi(1+it, A)x$. Then $y=(\int_{B(D)}\phi(1+it, z)(z-A)^{-1}dz)x$. Thus there exists a sequence of partitions P_n of $B(D)$ with "norms" converging to 0, and a corresponding sequence of "Riemann sums",

$$S_n = \sum_{k=1}^{n}\phi(1+it, z_k')(z_k' - A)^{-1}(z_k - z_{k-1})$$

satisfying S_nx converges to y as n approaches infinity.

One observes that for each x in X, S_nx is an element of $D(A)$. The operator A by assumption is closed in X. Thus, to establish that y is a member of $D(A)$, it is sufficient to show that the sequence $A(S_nx)$ has a limit as n approaches infinity. We presently obtain this result. Recall that $B(D)$, the boundary of D is compact. Thus the set,

$$\{\,|z|\mid z \text{ is a member of } B(D)\}$$

has a maximum value which we denote by "T". Now let $z\in B(D)$. Then

$$\begin{aligned}\|A(z-A)^{-1}\| &= \|((A-z)+z)(z-A)^{-1}\|\\ &\leqq \|(A-z)(z-A)^{-1}\| + |z|\,\|(z-A)^{-1}\|\\ &= \|I\| + |z|\,\|(z-A)^{-1}\| \quad (I \text{ is the identity operator})\\ &\leqq 1+TM.\end{aligned}$$

Thus, for each z in $B(D)$, $A(z-A)^{-1}$ is a bounded linear operator on X. Furthermore, $A(z-A)^{-1}$ is continuous on $B(D)$, for let z, z_0 be members of $B(D)$. Then

$$\begin{aligned}\|A(z-A)^{-1}-A(z_0-A)^{-1}\| &= \|A((z-A)^{-1}-(z_0-A)^{-1})\|\\ &= \|A(z_0-z)(z-A)^{-1}(z_0-A)^{-1}\|\\ &\leqq |z_0-z|\,\|A(z-A)^{-1}\|\|(z_0-A)^{-1}\|\\ &\leqq |z_0-z|\,(1+TM)M.\end{aligned}$$

By assumption (b), part (2) $\phi(1+it, z)$ is an analytic function of z in a neighborhood of the closure of D. Hence, it is certainly continuous on $B(D)$. Thus $\int_{B(D)}\phi(1+it,z)\,A(z-A)^{-1}dz$ exists. Consequently,

$$A(S_nx) = \left\{\sum_{k=1}^{n}\phi(1+it, z_k')A(z_k'-A)^{-1}(z_k-z_{k-1})\right\}x$$

converges to $\{\int_{B(D)}\phi(1+it, z)A(z-A)^{-1}dz\}x$ as n approaches infinity. Thus, $y\in D(A)$.

One now observes that

$$\begin{aligned}\|\phi(1+it, A)x\| &= \left\|\left\{\int_{B(D)}\phi(1+it, z)(z-A)^{-1}dz\right\}x\right\|\\ &\leqq \left\{\int_{B(D)}|\phi(1+it, z)|\,\|(z-A)^{-1}\|\,|dz|\right\}\|x\|\\ &\leqq \left\{\int_{B(D)}K_1\|(z-A)^{-1}\|\,|dz|\right\}\|x\|\end{aligned}$$

(by assumption (b), part (4))

$$\leqq K_1ML\|x\|.$$

Thus,

$$\begin{aligned}\|\phi(1+it, A)x\|_{D(A)} &= \|\phi(1+it, A)x\| + \|A(\phi(1+it, A)x)\| \\ &= \left\| \left\{ \int_{B(D)} \phi(1+it, z)(z-A)^{-1} dz \right\} x \right\| \\ &\quad + \left\| \left\{ \int_{B(D)} \phi(1+it, z) A(z-A)^{-1} dz \right\} x \right\| \\ &\leqq K_1 ML\|x\| + K_1(1+TM)L\|x\| \\ &= \{K_1 ML + K_1(1+TM)L\}\|x\| \\ &= \{K_1 L(1+M+TM)\}\|x\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\phi(it, A)x\| &= \left\| \left\{ \int_{B(D)} \phi(it, z)(z-A)^{-1} dz \right\} x \right\| \\ &\leqq \left\{ \int_{B(D)} |\phi(it, z)| \, \|(z-A)^{-1}\| \, |dz| \right\} \|x\| \\ &\leqq \left\{ \int_{B(D)} K_0 \|(z-A)^{-1}\| \, |dz| \right\} \|x\| \\ &\qquad\qquad \text{(by assumption (b), part (4))} \\ &\leqq K_0 ML\|x\|.\end{aligned}$$

Thus, $\phi(w, A)x$ is a member of $H(X, D(A))$, and

$$\begin{aligned}\|\phi(w, A)x\|_{H(X, D(A))} &= \max\left\{ \sup_t \|\phi(it, A)x\|, \sup_t \|\phi(1+it, A)x\|_{D(A)} \right\} \\ &\leqq \max\{(K_0 ML)\|x\|, K_1 L(1+M+TM)\|x\|\} \\ &= \max\{K_0 ML, K_1 L(1+M+TM)\}\|x\|.\end{aligned}$$

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