BOUNDED IN THE MEAN SOLUTIONS OF $\Delta u = Pu$
ON RIEMANNIAN MANIFOLDS

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Abstract. Let $\Phi$ be a convex positive increasing function and $d = \lim_{t \to \infty} \Phi(t)/t$. A harmonic function $u$ on a Riemann surface $R$ is called $\Phi$-bounded if $\Phi(|u|)$ is majorized by a harmonic function on $R$. M. Parreau has shown that if $d < \infty$ ($d = \infty$, resp.), then every positive (bounded, resp.) harmonic function on $R$ reduces to a constant if and only if every $\Phi$-bounded harmonic function does. In this paper analogues of these results are given for the equation $\Delta u = Pu$ ($P \geq 0$) on a Riemannian manifold.

Let $R$ be a noncompact orientable Riemannian $n$-manifold, $P$ a nonnegative smooth $n$-form on $R$ and $\Phi$ a convex positive increasing function on $[0, \infty)$ with $\Phi(0) = 0$. A solution $u$ of $\Delta u = Pu$ is said to be $\Phi$-bounded or bounded in the mean on $R$ if $\Phi(|u|)$ has a solution majorant on $R$. Set $d = \lim_{t \to \infty} \Phi(t)/t$. Denote by $PX(R)$ the subset of solutions of $\Delta u = Pu$ on $R$ satisfying a boundedness property $X$; examples of $X$ are $N$ (nonnegative), $B$ (bounded), $\Phi$ ($\Phi$-bounded). The maximum number of linearly independent functions in $PX(R)$ will be denoted by $\dim PX(R)$ even though $PX(R)$ in general is not a vector space. Consider the following statements:

(i) If $d < \infty$, then $\dim PN(R) = i$ if and only if $\dim P\Phi(R) = i$.

(ii) If $d = \infty$, then $\dim PB(R) = i$ if and only if $\dim P\Phi(R) = i$.

Parreau [6] established the validity of (i) and (ii) in the case $P \equiv 0$ and $i = 1$, that is for harmonic functions. In this paper we consider the case $P \neq 0$. If $P \neq 0$, the constants are not solutions of $\Delta u = Pu$ and therefore degeneracy of a Riemannian manifold $R$ with respect to a property $X$ has been taken to be $\dim PX(R) = 0$. In view of Myrberg's [3] result that $\dim PN(R)$ is always greater than zero and the fact that when $\dim PB(R) \geq 1$ a one dimensional subspace plays virtually the same role as the constants do in the space of bounded harmonic functions, we shall consider two degrees of degeneracy $\dim PX(R) = i$, $i = 0$ and 1.

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Our approach invokes some recent results in the axiomatic theory of harmonic functions. In particular we begin by showing that an easy consequence of Loeb-Walsh [3] is the two domain criterion [1, p. 213], well known for harmonic functions on Riemann surfaces. Nakai [5] has shown that Parreau's results are valid for almost arbitrary nonnegative real-valued functions \( \Phi \) but in our extension to \( P \neq 0 \) we have been unable to relax the original requirements on \( \Phi \).

1. We first introduce some terminology and results from Loeb-Walsh [2]. Let \( \mathcal{C} \) be a harmonic class on \( W \) satisfying Loeb's Axiom IV, i.e. \( 1 \in \mathfrak{C} \). Let \( e \) be the greatest \( \mathcal{C} \)-harmonic minorant of \( 1 \) and assume \( e \neq 0 \). It is also assumed that the Banach lattice \( \mathcal{B} \mathcal{C} \) of bounded functions in \( \mathcal{C} \) does not reduce to the constants. (Note that in the case \( \Delta u = Pu, \ P \neq 0 \), this already follows from the fact that \( e \neq 1 \).) Let \( W^* \) be the \( \mathcal{B} \mathcal{C} \)-compactification of \( W \), \( \Gamma = W^* - W \) and \( \Delta = \{ t \in \Gamma | e(t) = 1 \} \) and \( f \wedge \infty g(t) = f \wedge g(t) \) for every \( f, g \in \mathcal{B} \mathcal{C} \).

The symbols \( A, bA \) will denote the closure and boundary of \( A \) with respect to \( \overline{W} = W \cup \Delta \) and by \( \partial A \) we mean \( bA \cap W \). By 1.1 and 2.1 of [2] we see that for any region \( \Omega \subset W \), \( b \Omega \) is associated to \( \overline{\mathfrak{C}} \) (i.e. if \( s \in \mathfrak{C} \), is bounded from below and \( \lim s \geq 0 \) at \( b \Omega \), then \( s \geq 0 \)). Theorem 2.3 of [2] states that the restriction mapping gives an isometric isomorphism of \( \mathcal{B} \mathcal{C} \) onto \( C(A) \) which preserves positivity and the lattice operations. From 1.2 and 2.5 of [2] we see that every point of \( \Delta \) is regular for the Dirichlet problem.

We now introduce the

**Definition.** A region \( \Omega \subset W \) is in the class \( SO_{bx} \) if \( \partial \Omega \) is regular for the Dirichlet problem and there exists no nonzero element of \( \mathcal{B} \mathcal{C} \) which vanishes continuously on \( \partial \Omega \).

As a consequence we have the

**Theorem.** There exist at least \( k (\geq 1) \) pairwise disjoint regions \( \Omega_i \subset W \), \( \partial \Omega_i \) regular and \( \Omega_i \in SO_{bx} \) if and only if \( \dim \mathcal{B} \mathcal{C} \geq k \).

If \( \Omega_i \in SO_{bx} \), then \( b \Omega_i - \partial \Omega_i \neq \emptyset \) because we know \( b \Omega_i \) is associated to \( \overline{\mathfrak{C}} \). Thus the existence of \( k \) such \( \Omega_i \)'s means that \( \Delta \) contains at least \( k \) points and hence \( \dim \mathcal{B} \mathcal{C} = \dim C(\Delta) \geq k \). Conversely suppose \( \dim \mathcal{B} \mathcal{C} \geq k \). Then there exist \( k \) points \( \bar{p}_j \in \Delta \) and \( u_i \in \mathcal{B} \mathcal{C}, 0 \leq u_i \leq 1 \) with the property \( u_i(p_j) = \delta_{ij}, 1 \leq i, j \leq k \). Set \( \Omega_i = \{ x \in W | u_i > \max(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k) \} \). If \( x_0 \in \partial \Omega_i \), then there is a \( u_i \) such that \( u_i(x_0) = u_i(x_0) \) and consequently \( u_i - u_1 \) is a barrier for \( \Omega_i \) at \( x_0 \) Thus \( \partial \Omega_i \) is regular and, in view of the regularity of \( \Delta \), \( b \Omega_i \) is regular. Since \( b \Omega_i - \partial \Omega_i \neq \emptyset \) we can find \( h \in \mathcal{B} \mathcal{C} \) such that \( h \neq 0 \) and \( h|\partial \Omega_i = 0 \), i.e. \( \Omega_i \in SO_{bx} \).
2. We now turn to the harmonic class determined by the solutions of $\Delta u = Pu$ on a Riemannian manifold $R$.

**Theorem.** If $P \neq 0$, then (i) is valid for $i = 0$ and 1.

If $d < \infty$, then there is a $c > 0$ such that $\Phi(t) \leq ct$, $t \geq 0$. By Myrberg's result we always have a nonzero solution $u \in P\mathcal{N}(R)$ and since $\Phi(u) \leq cu$, $u \in P\Phi(R)$. Thus (i) holds for $i = 0$.

Suppose $\dim P\mathcal{N}(R) > 1$ then by the above observation we also have $\dim P\Phi(R) > 1$. That $\dim P\Phi(R) > 1$ implies $\dim P\mathcal{N}(R) > 1$ is a direct consequence of a result of Nairn [4, Corollary 10.2] to the effect that every $\Phi$-bounded solution is the difference of two nonnegative ones.

3. Another result of Nairn [4, Lemma 9] is that if $u$ is a solution, then $\Phi(|u|)$ is a subsolution. From this we obtain the

**Lemma.** $P\mathcal{B}(\Omega) \subset P\Phi(\Omega)$, for any $\Phi$.

In fact $\Phi(|u|)$ is a bounded subsolution for a function $u \in P\mathcal{B}(\Omega)$. The family $F$ of all supersolutions greater than $\Phi(|u|)$ is a nonempty Perron family. Thus the pointwise inf $v$ of $F$ is a solution with $\Phi(|v|) \leq v$.

4. As in §1 we denote by $SO_{px}$ the subregions $\Omega$ of $R$ such that $\partial \Omega$ is regular for the Dirichlet problem and $PX(\Omega)$ contains no nonzero element which vanishes continuously on $\partial \Omega$. The remainder of our result is based on the following (cf. [1, p. 218]).

**Theorem.** If $d = \infty$, then $SO_{px} = SO_{pb}$, otherwise $SO_{px} = SO_{PN}$.

Myrberg's [3] argument can be used to show that $SO_{PN} = \emptyset$. Take $\Omega \subset R$ such that $\partial \Omega$ is regular and $\{R_n\}$ is an exhaustion of $R$, with $\partial R_n$ regular and $\bar{R}_n$ compact in $R$. Fix $x \in \Omega \cap R_1$ and let $\omega_n$ be a solution on $\Omega \cap R_n$ such that $\omega_n|\partial \Omega \cap R_n = 0$, $\omega_n|\partial R_n \cap \Omega = c_n$ (const.) determined by $\omega_n(x) = 1$. From the sequence $\{\omega_n\}$ of positive solutions we can extract a subsequence which converges to a nonnegative solution $\omega$ uniformly on compact subsets of $\Omega \cup \partial \Omega$. Since $\omega(x) = 1$, $\omega|\partial \Omega = 0$, we have $\omega \in SO_{PN}$. Thus $d < \infty$ implies as before that $SO_{px} = \emptyset$.

That $SO_{PB} \subset SO_{PB}$ follows from Lemma 3 and to complete the proof we show $SO_{PB} \subset SO_{PB}$, if $d = \infty$. To this end suppose $\Omega \in SO_{PB}$, that is $\partial \Omega$ is regular and there exist solutions $u, v$ on $\Omega$ such that $\Phi(|u|) \leq v$ and $u$ vanishes continuously on $\partial \Omega$. Set $w_n = \omega_n/c_n$ and extract a subsequence again denoted by $\{w_n\}$ which converges to a solution $w$ uniformly on compact subsets of $\Omega \cup \partial \Omega$. We want to show that for the fixed $x \in \Omega \cap R_1$, $w(x) \neq 0$.
We let $I$ be the positive bounded linear functional which assigns to each $f \in C(\partial(R_n \cap \Omega))$ the solution to the Dirichlet problem with boundary values $f$ evaluated at $x$. By Jensen's inequality applied to the integral representation of $I$ we have

$$\Phi(I(\{ u \mid w_n \})/I(w_n)) \leq (I(\Phi(\{ u \mid w_n \})/I(w_n)).$$

But $I(\{ u \mid w_n \}) = I(\{ u \mid x \}) \geq |u(x)|$ and $I(\Phi(\{ u \mid w_n \}) = I(v) = v(x)$. Thus

$$\Phi(\{ u(x) \mid w_n(x) \}) \leq v(x)/w_n(x) = (v(x)/w_n(x))(\{ u(x) \mid w_n(x) \}).$$

Since $d = \infty$, the above implies that $\{ |u(x)|/w_n(x) \}$ is bounded, i.e. $w_n(x) \rightarrow w(x) > 0$.

5. We now establish the

**Theorem.** If $P \neq 0$, then (ii) is valid for $i = 0$ and 1.

Indeed Lemma 3 gives $\dim PB(R) > i$ implies that $\dim P\Phi(R) > i$. It remains to show that $\dim P\Phi(R) > i$ implies that $\dim PB(R) > i$, $i = 0, 1$ for then a simple logical manipulation gives the assertion.

If $\dim P\Phi(R) > 1$, then there exists a $\Phi$-bounded solution $v$ which changes sign. Indeed if $v_1$ and $v_2$ are linearly independent $\Phi$-bounded solutions that do not change sign, then we assume they are of opposite sign. We fix $x \in R$ and $\lambda \in (0, 1)$ such that $\lambda v_1(x) + (1 - \lambda)v_2(x) = 0$. By the convexity of $\Phi$ the solution $v = \lambda v_1 + (1 - \lambda)v_2$ is $\Phi$-bounded and must change sign by the Harnack principle. Set $\Omega_1 = \{ v < 0 \}$ and $\Omega_2 = \{ v > 0 \}$. In view of Theorem 4 and Theorem 1 we have $\dim PB(R) > 1$.

To complete the proof we need only show that $\dim PB(R) > 0$ follows from the assumption that $\dim P\Phi(R) = 1$. The result of Naïm cited above on $\Phi$-bounded solutions being the difference of nonnegative ones allows us to assert the existence of $v$, a positive $\Phi$-bounded solution on $R$. By Sard's theorem there is a $c > \inf v$ such that the hypersurface $v^{-1}(c)$ is smooth. Thus there is a bounded solution $v'$ on a component $\Omega$ of the set $\{ v > c \}$ such that $v \geq v' \geq 0$ and $v'$ has continuous boundary value $c$ on $\partial\Omega$. Since $\Phi(v - v') \leq \Phi(v)$ we again apply Theorem 4 and Theorem 1 to conclude that $\dim PB(R) > 0$.

**References**


