ON THE COMPLETENESS OF HAMILTONIAN VECTOR FIELDS

WILLIAM B. GORDON

ABSTRACT. Sufficient conditions are given for a hamiltonian vector field to be complete which involve bounds on the potential or its gradient.

1. A vector field ξ on a manifold M is said to be complete iff for every $m \in M$ the maximal interval of existence (ω_-, ω_+) of every solution of

$$dx/dt = \xi, \qquad x(0) = m, \tag{1}$$

is given by $\omega_\pm = \pm\infty$, cf. [1]. Hence if ξ is of class C^1, so that solutions are unique, to say that ξ is complete means that ξ generates a 1-parameter group or (global) flow on M. Let (t_-, t_+) be a bounded open neighborhood of 0 in which a solution $x = x(t)$ to (1) is defined. It is well known that $\omega_\pm = \pm\infty$ iff $x(t)$ remains in a compact set as t varies over any such neighborhood, [2]. A device which assures this property is provided by the following simple lemma.

LEMMA. *Let ξ be a C^0 vector field on a manifold M of class C^1. Then ξ is complete if there exist a C^1 function E, a proper C^0 function f, and constants α, β such that for all $m \in M$*

(i) $|\xi E(m)| \leqq \alpha |E(m)|$,

(ii) $|f(m)| \leqq \beta |E(m)|$.

(Recall that f proper means f^{-1} (compact) $=$ compact.)

PROOF. From basic definitions $\xi E(m) = dE(x(t))/dt|_{t=0}$. Hence from Gronwall's inequality, or otherwise, it follows that $|E(x(t))| \leqq |E(x(0))| e^{\alpha|t|}$, $\omega_- < t < \omega_+$ so that $|f(x(t))| \leqq \beta |E(x(0))| e^{\alpha|t|}$. Since f is proper, this means that $x(t)$ remains in a compact set as t varies over a bounded neighborhood of 0 (for which a solution is defined).

2. To apply this lemma to the case $\xi =$ hamiltonian vector field, let (M, g) be a riemannian manifold of class C^1 with metric tensor g, and let (q, p) denote local coordinates on $T^*(M)$. Every C^1 "potential" V on M gives rise to a hamiltonian $H = T + V =$ kinetic energy $+$ potential $= \frac{1}{2}\sum g^{ij}p_ip_j + V(q)$ whose corresponding hamiltonian vector

Received by the editors November 1, 1969.

AMS 1969 *subject classifications.* Primary 3465; Secondary 3404.

Key words and phrases. Complete vector field, hamiltonian vector field.

field ξ on $T^*(M)$ is given by $\xi = \sum\{(\partial H/\partial p_i)\partial/\partial q_i - (\partial H/\partial q_i)\partial/\partial p_i\}$. We now prove the following

THEOREM. *Let (M, g), ξ, V be as above. Then ξ is complete if any of the following are true.*

(i) *V is proper and bounded below, say $V \geqq 0$.*

(ii) *(M, g) is complete (in the riemannian sense) and $V \geqq 0$.*

(iii) *(M, g) is complete and $\|\nabla V\|$ is bounded.*

(iv) *(M, g) is complete and $\|\nabla V\| \leqq$ constant $\cdot \|w\|$ where $m \to w(m)$ is an isometric embedding of M into euclidean space.*

PROOF. For (i) apply the lemma with $f = E = H$. Since V is a proper function on M, and $V \geqq 0$, it follows that H is proper on $T^*(M)$. But $\xi H = 0$, so that the hypotheses of the lemma are satisfied. For the remainder of the theorem, let $q \to w(q)$ be an isometric embedding of M into euclidean space R^n. (The existence of such embeddings is given by a theorem of Nash [3].) Since (M, g) is complete as a riemannian manifold, M is complete with respect to the metric induced by g. Therefore M is a closed submanifold of R^n. It follows that $r^2(m) = \|w(m)\|^2$ is a proper function on M. Also we have

$$(2) \qquad |\xi r^2| = 2|\langle w, \sum p_i w^i\rangle| \leqq 2\|w\| \cdot \|\sum p_i w^i\| \leqq 2\|w\| \cdot (2T)^{1/2}$$

where $\langle\ ,\ \rangle$ is the standard inner product on R^n, $w_i = \partial w/\partial q_i$ and $w^i = \sum g^{ij} w_j$, so that $g_{ij} = \langle w_i, w_j\rangle$ and $g^{ij} = \langle w^i, w^j\rangle$. To prove (ii), set $E = H + r^2$. Then it is easy to show that $|\xi E/E| \leqq 2$. To obtain a function f satisfying the hypotheses of the lemma, set $f = r^2$ (or $f = E$). To prove (iii) and (iv) let $f = E = T + \frac{1}{2}r^2$. A direct calculation shows that $|\xi T| = |\sum g^{ij} p_i \partial V/\partial q_i| \leqq (2T)^{1/2} \cdot \|\nabla V\|$ so that $|\xi E/E| \leqq (2T)^{1/2}\|w\|(1 + \|\nabla V\|/\|w\|)/(T + \frac{1}{2}\|w\|^2)$. The proof of (iv) is now immediate. To prove (iii) we need only choose an embedding for which $\|w(m)\|$ is bounded below by a positive number.

Note that (i) has as a consequence the well-known fact that every hamiltonian vector field attached to (the cotangent bundle of) a compact manifold is complete, this being a generalization of the fact that every compact riemannian manifold is complete in the riemannian sense, (the case $V = 0$).

Finally, we remark that the lemma and the theorem can easily be extended to the nonautonomous case $\xi = \xi(m, t)$ by the usual device: one considers the vector field $\hat{\xi} = \xi + \partial/\partial t$ on $M \times R$ (in the lemma) and $T^*(M) \times R$ (in the theorem) to obtain sufficient conditions for completeness. For example, parts (iii) and (iv) of the theorem remain true if the potential is time dependent; to see this replace the function E in the argument by $\hat{E} = E + t^2$. Then $\hat{E}$ is a proper function on $T^*(M)\times R$ and $|\hat{\xi}\hat{E}/\hat{E}| \leqq$ constant. On the other hand, generalizations of parts (i) and (ii) apparently require boundedness conditions on $\partial V/\partial t$.

BIBLIOGRAPHY

1. R. Abraham, *Foundations of mechanics*, Benjamin, New York, 1967. MR **36** #3527.

2. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

3. J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63. MR **17**, 782.

NAVAL RESEARCH LABORATORY, CODE 7840, WASHINGTON, D. C. 20390