MEASURES OF N-FOLD SYMMETRY
FOR CONVEX SETS

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Abstract. If a convex set $S$ is 3-fold symmetric about a point $0 \in S$, then any 3-star contained in $S$ with vertex 0 is no smaller than any other parallel 3-star contained in $S$. In this paper, among other results, we establish the converse. Consequently, we find two measures of $n$-fold symmetry, one for $n = 2, 3$, and the other for each $n \geq 2$.

1. Introduction. A set $S$ in the complex plane $\mathbb{C}$ is said to be $n$-fold symmetric about a point $P$ if every rotation about $P$ of $2\pi/n$ maps $S$ onto itself. Let $\mathcal{F}$ be the family of all compact convex sets in $\mathbb{C}$ with nonempty interior, that is, different from a line segment. Following Grünbaum [2], we call a real-valued function $F$ a similarity invariant measure of $n$-fold symmetry for $\mathcal{F}$, if

(i) $0 \leq F(K) \leq 1$, $K \in \mathcal{F}$;
(ii) $F(K) = 1$ if and only if $K \in \mathcal{F}$ has a center of $n$-fold symmetry;
(iii) $F(K) = F(T(K))$ for every $K \in \mathcal{F}$ and every nonsingular similarity transformation $T$ of $\mathbb{C}$ onto itself;
(iv) $F$ is continuous on $\mathcal{F}$.

There is a large literature on such measures in case $n = 2$ (cf. [2]). In this paper we are going to introduce two definitions of measures for other natural numbers $n$. [See §3.] Our first definition, which holds for $n = 2, 3$, comes from Theorem A. This theorem also gives a partial answer to a conjecture in the second author’s dissertation [4]. Our second definition, which holds for all $n \geq 2$, is based on Theorem B, a generalization of Hammer’s result [3].

We shall use the same terminology as introduced in [5]. Let $S$ be a convex set in $\mathbb{C}$, $0 \in S$ and let $T$ be an arbitrary $n$-star at 0 contained in $S$. That is, $T$ is any $n$-star with vertex 0 and rays terminating on the boundary of $S$. Then $S$ is said to have the $n$-maximal property at 0 if every $n$-star $U$, contained in $S$ and parallel to $T$, is no larger than $T$: $|U| \leq |T|$. The second author [4], [5] proved that every bounded convex set $S$, $n$-fold symmetric about $0 \in S$, has the $n$-
maximal property at 0 for every natural number $n \geq 2$. He also conjectures that if a convex set $S$ has the $p$-maximal property at a point $0 \in S$, where $p$ is a prime, then $S$ should be $p$-fold symmetric about 0. It is easy to see that the condition $p$ being a prime cannot be omitted. For if $S$ has the $n$-maximal property at 0, it also has the $mn$-maximal property at 0 for every natural number $m$. In particular, every rectangle has the 4-maximal property at its center 0, although it is only 2-fold symmetric about 0. The above conjecture is contained in the following:

**Conjecture.** Let $S$ be a bounded convex set in $C$ with the $n$-maximal property at $0 \in S$, and let $n = p_1 \cdots p_k$, where $p_1, \cdots, p_k$ are primes. Then $S$ is $p_j$-fold symmetric about 0 for some $j = 1, \cdots, k$.

The following theorem gives a partial answer to this conjecture.

**Theorem A.** Let $G$ be a bounded convex set in $F$ with the $n$-maximal property at $0 \in G$. Then if $n = 2, 3$, $G$ is $n$-fold symmetric about 0, and if $n = 4$, $G$ is 2-fold symmetric about 0.

The result for $n = 2$ is not new, and easily follows from a result of Hammer [3].

Let $S$ be a bounded convex set in $C$ containing the origin 0 and let $\rho(\theta) = \rho(\theta + 2\pi)$, $-\infty < \theta < \infty$, be the polar representation of the boundary of $S$. We say that $S$ has the $n$-supporting-line property with respect to 0, if for every $\theta$, there are $n$ lines of support of $S$ at the angles of $\theta, \theta + 2\pi/n, \cdots, \theta + 2(n - 1)/n$, such that the angle between any two adjacent lines is $(n - 2)\pi/n$. It is clear that if $S$ is $n$-fold symmetric about 0, it has the $n$-supporting-line property with respect to 0. We will prove the converse, namely

**Theorem B.** Let $n \geq 2$ be an arbitrary natural number and let $G$ be a bounded convex set in $F$ having the $n$-supporting-line property with respect to an interior point 0 of $G$. Then $G$ is $n$-fold symmetric about 0.

2. **Proofs of Theorems A and B.** We first prove Theorem B. Let $C$ be the boundary curve of $G$ with polar representation $\rho = \rho(\theta)$, and let $\psi = \psi(\theta)$ denote the angle at $(\theta, \rho)$ measured from the radial line to the line of support in the counterclockwise direction. Since $G$ is convex, $C$ has continuous turning tangents at all but a countable number of points, and by elementary calculus, we know that at these points

$$\cot \psi(\theta) = \rho'(\theta)/\rho(\theta).$$
The $n$-supporting-line property implies

$$\psi(\theta) = \psi(\theta + 2\pi/n)$$

for all $\theta$. Hence, integrating $\rho'/\rho$ and taking exponentials, we see that there exists a positive constant $c$ such that

$$\rho(\theta) = c_0(\theta + 2\pi/n) = \cdots = c^{n-1}(\theta + 2(n-1)\pi/n) = c^n\rho(\theta).$$

Hence, $c = 1$ and $G$ is $n$-fold symmetric about 0.

We shall prove Theorem A for $n = 3$. The proofs for $n = 2, 4$ are similar and easier. We first list the following five geometric observations, the proofs of which are quite elementary.

1. Let $A$ be an equilateral triangle with center 0 and interior $G$. Let $T$ be a 3-star contained in $G$ with vertex 0, and let $U$ be any 3-star contained in $G$ and parallel to $T$. Then $|U| \leq |T|$ and equality holds if and only if either the rays of $U$ fall on different sides of $A$ or $U = T$.

2. Let $T$ be a 3-star of finite length, and let $r_1, r_2, r_3$ be its rays. Let $U$ be another 3-star parallel to $T$ with rays $r_1, r_2, r_3$ such that the vertex of $U$ lies on $r_3$; $r_1, r_2, r_3, r_3$ terminate on a common straight line; and $r_3, r_3$ terminate at the same point. Then $|T| \leq |U|$, and equality holds if and only if $T = U$.

3. Let $T$ be a 3-star with rays $r_1, r_2, r_3$ and let $r_1, r_2$ terminate on straight lines $L_1, L_2$ respectively, where either $L_1$ and $L_2$ are parallel or they intersect on the same side of the line passing through the tips of $r_1, r_2$ as $r_3$. Then it is possible to construct a 3-star $U$ with rays $r_1, r_2, r_3$ parallel to $r_1, r_2, r_3$ respectively, such that $r_j$ terminates on $L_j$, $j = 1, 2$; $r_3$ and $r_3$ terminate at the same point; $r_3 \subset r_3$; and $|T| < |U|$.

4. Let $T$ be a 3-star contained in a triangle $\Delta$ such that the vertex of $T$ lies in the interior of $\Delta$ and each ray of $T$ terminates at an interior point of a different side of $\Delta$. Construct the three equilateral triangles $\Delta_1, \Delta_2, \Delta_3$ such that one side of each $\Delta_j$ lies on an extended side of $\Delta$ and the other two sides of $\Delta_j$ pass through the other two tips of the rays of $T$. Then at least one of $\Delta_1, \Delta_2, \Delta_3$ contains $T$ in its interior.

5. Let $\Delta$ be a triangle which is not equilateral, and let $T$ be a 3-star contained in $\Delta$ such that the vertex of $T$ lies in the interior of $\Delta$ and each ray of $T$ terminates at an interior point of a different side of $\Delta$. Then there is a 3-star $U$ contained in $\Delta$ and parallel to $T$ such that the vertex of $U$ is arbitrarily close to that of $T$ and $|U| > |T|$.

The proofs of (1) through (4) are quite straightforward, while the proof of (5) follows from (1) through (4). As a corollary of these observations, we obtain
Theorem 2.1. Let \( G \) be the interior of a convex polygon \( P \), \( T \) a 3-star with vertex in \( G \) and contained in \( G \) such that \( |T| \geq |U| \) where \( U \) is any 3-star contained in \( G \) and parallel to \( T \). Then either at least one ray of \( T \) terminates at a vertex of \( P \) or the three rays of \( T \) terminate on the interior of three different sides of \( P \), which, when extended, form an equilateral triangle that contains \( T \) in its interior.

Proof. Suppose that the three rays of \( T \) terminate on the interior of the sides of \( P \). By (2) these rays terminate on different sides, and by (3) these three sides, when extended, form a triangle which contains \( G \) in its interior. By (5) this triangle is indeed equilateral.

We remark that for the special case of convex polygons Theorem A is already proved. Indeed, from the above proof we see that 3-maximal property implies 3-supporting-line property which in turn, by Theorem B, implies 3-fold symmetry. To include a larger collection of convex sets, we need the following

Lemma 2.1. Let \( G \) be a convex domain with boundary \( C \). Let \( T \) be a 3-star with vertex \( 0 \in G \) and rays terminating on \( C \) at points where \( C \) has continuous turning tangents, such that \( T \) is no smaller than any parallel 3-star contained in \( G \). Then the lines of support \( L_1, L_2, L_3 \) of \( G \) at the tips \( a_1, a_2, a_3 \), respectively, of \( T \) form an equilateral triangle.

Proof. The lines of support form one of the configurations as described in our above five observations. For instance, assume that they form a triangle, which is not equilateral, such that \( T \) terminates on the interior of its three different sides as in observation (4). Let \( L_1 = L'_1 \) be the line of support on which an equilateral triangle as described in the conclusion of (4) can be constructed. Let \( L'_2 \) and \( L'_3 \) be drawn through \( a_2 \) and \( a_3 \) respectively to form this equilateral triangle. One can find a 3-star \( T^* \) contained in \( G \), parallel to \( T \) and with vertex \( 0^* \in G \) arbitrarily close to \( 0 \) so that the rays of \( T^* \) intersect \( L'_2 \) and \( L'_3 \) in \( G \) or on \( C \). [Cf. Fig. a or Fig. b.] Using the fact that a tangent line to a curve is a much closer approximation to the curve than any secant line, one can easily see that if \( 0^* \) is suitably chosen \( |T^*| > |T| \). This is a contradiction. The proofs for the other configurations are similar.

We can now complete the proof of Theorem A \( (n = 3) \). Let \( C \) be the boundary curve of \( G \). Then \( C \) has continuous turning tangents at all but a countable number of points. Hence, by using the right-hand derivatives and Lemma 2.1, \( G \) has the 3-supporting-line property with respect to \( 0 \), and is, therefore, 3-fold symmetric about \( 0 \) by Theorem B.
3. Measures of symmetry. Let $G \in \mathfrak{S}$, $P$ a point in $G$ and let $P_\theta$, $0 \leq \theta < 2\pi/n$, denote the $n$-star contained in $G$, with vertex $P$, and having a ray with an angle of inclination $\theta$ measured positively from the real axis. Let

$$M_n(G; P) = \inf_{Q, \theta} \frac{|P_\theta|}{|Q_\theta|},$$

where $Q$ runs over $G$ and $0 \leq \theta < 2\pi/n$. We define the function $M_n$ on $\mathfrak{S}$ by

$$M_n(G) = \sup_{P \in G} M_n(G; P), \quad G \in \mathfrak{S}.$$

**Theorem 3.1.** For $n = 2, 3$, $M_n$ is a similarity invariant measure of $n$-fold symmetry for $\mathfrak{S}$.

**Proof.** It is clear that $M_n$ satisfies (i) in §1. Since each $G \in \mathfrak{S}$ has nonempty interior, the useful $n$-stars $Q_\theta$ in the definition of $M_n(G; P)$ have lengths bounded away from zero; hence, $M_n$ is continuous. Finally, for $n = 2, 3$, we see that, using Theorem A, $M_2(G) = 1, G \in \mathfrak{S}$, if and only if $G$ is $n$-fold symmetric.

Note that in defining $M_n$, we have required that the sets $G$ to have nonempty interior, that is, to be different from a line segment. In the latter case, it is not clear how to interpret $|P_\theta|/|Q_\theta|$, when both $|P_\theta|$ and $|Q_\theta|$ are zero for all, except one, values of $\theta$, $0 \leq \theta < 2\pi/n$. Indeed, in this case, continuity for $M_3$ breaks down: If we approximate a line segment $L$ by rectangles, the limit would be $\frac{1}{2}$; however, if we approximate $L$ by isosceles triangles, it would be $2/5$.

We now define our second measure based on Theorem B. Let
$G \in \mathcal{F}$, $P$ a point in $G$ and $P_\theta$ as defined above, $0 \leq \theta < 2\pi/n$. Let $\Delta$ be a regular $n$-gon such that $G$ is contained in the closure of its interior and such that if $\Delta'$ is another regular $n$-gon which lies in the interior of $\Delta$, then $G$ does not lie in the closure of the interior of $\Delta'$. Such a $\Delta$ will be called admissible. Let $Q$ be the center of $\Delta$. (If $n=2$, $\Delta$ is a pair of parallel lines and $Q$ is equidistant from these two lines.) Let $Q_\theta$ be the $n$-star contained in the interior of $\Delta$ with vertex $Q$ and parallel to $P_\theta$. We define

$$N_n(G; P) = \inf_{\Delta} \sup_{\theta} \frac{|P_\theta|}{|Q_\theta|},$$

where $0 \leq \theta < 2\pi/n$ and $\Delta$ runs over all admissible regular $n$-gons. We now define the function $N_n$ on $\mathcal{F}$ by

$$N_n(G) = \sup_{P \in G} N_n(G; P), \quad G \in \mathcal{F}.$$

**Theorem 3.2.** For each natural number $n \geq 2$, $N_n$ is a similarity invariant measure of $n$-fold symmetry for $\mathcal{F}$.

**Proof.** For $G \in \mathcal{F}$, it is obvious that $0 \leq N_n(G) \leq 1$. If $G$ has a positive diameter $d$, each admissible $\Delta$ has diameter no less than $d$. Hence, the corresponding $Q_\theta$ has length bounded away from $d$. Continuity of $N_n$ follows. By Theorem B, it is not difficult to see that $N_n(G) = 1$, $G \in \mathcal{F}$, if and only if $G$ is $n$-fold symmetric.

Note that different from $M_n$ ($n=3$, say), the continuity of $N_n$ even holds for a line segment $L$. In fact, approximating $L$ by sets in $\mathcal{F}$, we should define $N_2(L) = 1$ and $N_3(L) = \frac{1}{3}$. Next, we find a lower bound of $M_1(G)$, $G \in \mathcal{F}$, larger than zero. We need the following lemma (cf. [1]).

**Lemma 3.1.** Every convex set of diameter $d$ is contained in a circle of diameter no greater than $2d/\sqrt{3}$.

**Theorem 3.3.** For each $G \in \mathcal{F}$, $1/2\sqrt{3} \leq N_3(G) \leq 1$.

**Proof.** For an admissible $\Delta$, let $Q_\theta$ be the corresponding 3-star parallel to $P_\theta$. Using Lemma 3.1 and elementary geometry, we see that

$$\sup_{\theta} \frac{|P_\theta|}{|Q_\theta|} \geq \frac{1}{2\sqrt{3}}$$

for every admissible $\Delta$. Hence, the result follows.
REFERENCES


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