HANDLEBODIES IN 3-MANIFOLDS

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ABSTRACT. Consider a knot, link or wedge of circles which bounds a singular surface of genus $g \geqq 0$ within an orientable 3-manifold. Then there is a handlebody in the manifold within which the 1-complex bounds a singular surface of genus g. In particular, a handlebody which is contractible in an orientable 3-manifold is contractible within another handlebody.

Let K be a compact 1-complex in the interior of an orientable 3-manifold M; for simplicity we shall assume K to be a knot, a link, or a wedge of circles with vertex P, although it will be clear that our arguments can be extended to any 1-complex. Let $f: S \to \operatorname{Int} M$ be a continuous map of a (compact, but not necessarily connected) orientable 2-manifold S of genus $g \geqq 0$, such that $f|\operatorname{Bd} S$ is a homeomorphism onto K (except that if K is a wedge of circles, then f maps one point on each boundary component to P, but is otherwise $1:1$).

By modification of f within a regular neighbourhood of $f(S)$ using the methods of [2], [4], [5], [6], we may assume that the singularities of f consist of a finite number of

(i) double curves, i.e., pairs of curves on S which are matched by f;

(ii) triple points, at which 3 pairs of double curves cross; and

(iii) branch points, common endpoints of one or more pairs of matched double curves.

Furthermore these singularities may be moved slightly by an ambient isotopy of M so that

(iv) no triple point or branch point occurs on Bd S,

(v) only a finite number of double points occur on Bd S, and

(vi) all double curves lie outside a regular neighbourhood of $f^{-1}(P)$, if K is a wedge of circles.

[1]Let C be an arc in S joining a point on the boundary of S to a point Q on some double curve and such that

(a) C lies in the interior of S except for one endpoint,

(b) Q is the only point of C which is a singular point of f, and

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[1] I am grateful to the referee for pointing out that the process of normalization of singularities which follows is precisely that used by Haken [8].

(c) neither Q nor Q', the other point of S such that $f(Q)=f(Q')$, is a triple point or a branch point.

Since $f(C)$ is an arc, its regular neighbourhood E is a 3-cell. $f^{-1}(E)$ consists of two discs, D_1 = a regular neighbourhood of C and D_2 = a regular neighbourhood of Q'. Since Q' is not a branch point, $f(D_2)$ is a disc in E. Let D^* be the disc on Bd E which together with $f(D_2)$ bounds a subcell of E containing $f(C)$, and choose a homeomorphism $h:f(D_2)\to D^*$ which is the identity on the common boundary circle. Now modify f to a map f' by

$$\begin{aligned} f'(x) &= f(x) && \text{if } x \notin D_2, \\ &= hf(x) && \text{if } x \in D_2. \end{aligned}$$

This modification can of course be achieved by an isotopy in M pushing $f(Q)$ along $f(C)$ and over K. The resulting singular curves of f' are just those of f except for the change shown in Figure 1.

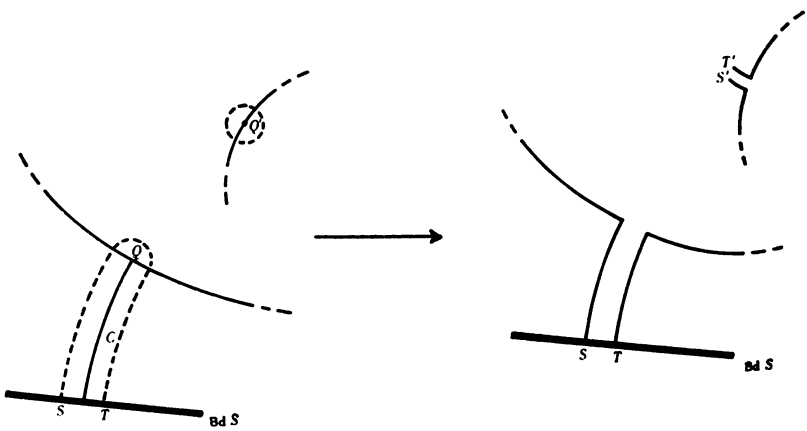

FIGURE 1

Using this method of "pushing double curves over the edge," we may modify f so that at least one member of each pair of double curves has an endpoint on Bd S. Furthermore (by choosing Q between points we wish to separate) we may assume that each double curve has at most one triple point or branch point. Note that no double curve is separated from the boundary of S by the other singularities.

Triple points are now easily taken care of by "pushing over the edge" if we choose for the arc C the double curve joining the triple

point to the boundary; the process is summarized in Figure 2. As we shall see, it is not necessary for our purpose to attempt to push the branch points over the edge.

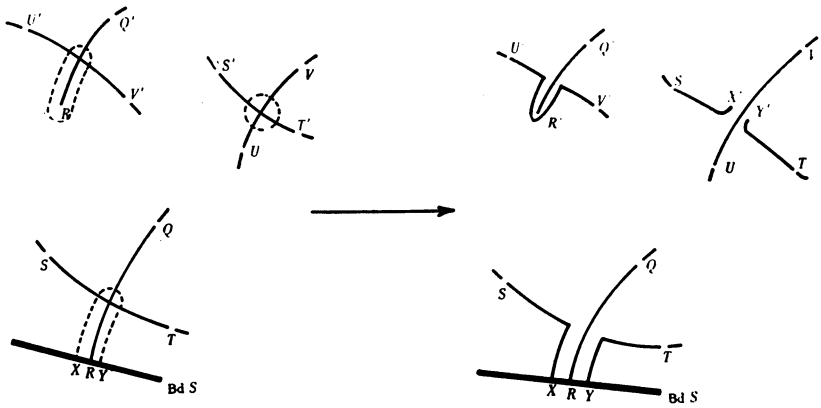

FIGURE 2

It is necessary, however, in the case of a branch point of multiplicity >1, to bring the curve of each matched pair joining the branch point to the boundary of S into a disc; this is shown in Figure 3. Finally, consider a matched pair of double curves C, C', the first having both endpoints on Bd S and the other lying entirely in the interior of S. Pushing C' over the edge, as shown in Figure 4, results in double curves all having exactly one endpoint on Bd S.

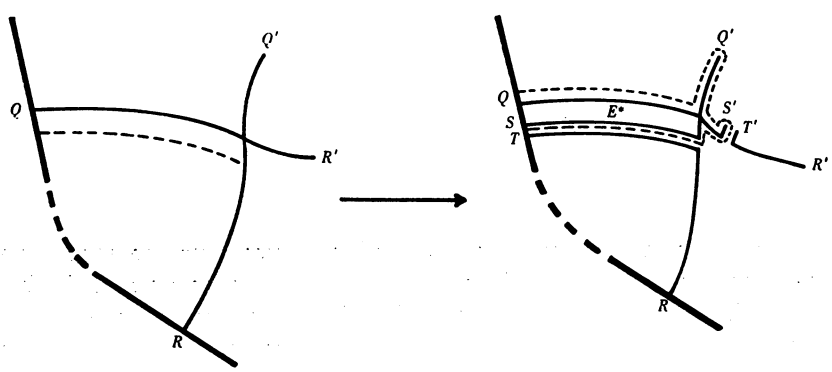

FIGURE 3

Thus the singularities of f may be assumed to be of 3 types:

(a) pairs of disjoint double arcs $C_1, C_1', \cdots, C_m, C_m'$ each leading from Bd S into the interior of S;

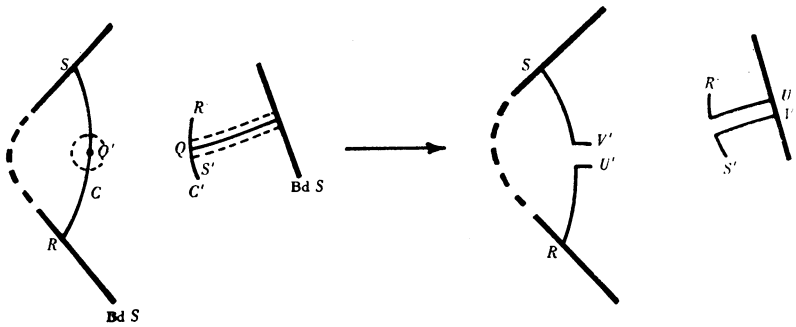

FIGURE 4

(b) trees of double curves from Bd S into S branching at one point, and contained in discs $E_1^*, \cdots, E_n^*$ in S each meeting Bd S in an arc; and

(c) the points of $f^{-1}(P)$, if K is a wedge of circles.

Now $f(C_i)=f(C_i')$ is an arc in M, $f(E_j^*)$ is contractible, and the singularities as listed have disjoint images in M, so there are 3-cell neighbourhoods in M whose inverse images are

(a) discs D_i, D_i', regular neighbourhoods of C_i, C_i' respectively in S,

(b) discs E_j, regular neighbourhoods of E_j^* in S, and

(c) discs F_k, regular neighbourhoods of the points $P_k \in f^{-1}(P)$.

The map f is 1:1 on the surface S', the closure of the complement of the union of all these discs; thus $f(S')$ has a regular neighbourhood N which is a disjoint union of handlebodies of total genus g (since M is orientable). In the case of a wedge of circles, a 3-cell neighbourhood of the point P meets N in its boundary in a regular neighbourhood of the arcs $\bigcup f(S' \cap F_k)$, i.e., in r disjoint discs; then N together with this cell forms a handlebody (of genus $\leqq g+r$). A 3-cell regular neighbourhood of each $f(E_j^*)$ meets N in its boundary in a regular neighbourhood of the arc $f(S' \cap E_j)$, that is, a disc. Adding these cells to N again gives a handlebody, or a disjoint union of handlebodies, of the same total genus. Finally a regular neighbourhood of each $f(C_i)$ meets N in its boundary in a regular neighbourhood of the two disjoint arcs $f(S' \cap D_i) \cup f(S' \cap D_i')$, that is, in a pair of discs. Thus adding these cells to N, and if necessary cells connecting the components of N, we have a handlebody (of genus $\leqq g+m+r$) which contains the singular surface $f(S)$. This proves the result stated in the abstract.

COROLLARY. *If V is a handlebody in the interior of an orientable 3-manifold M, and V is contractible (or nullhomologous) in M, then there is a handlebody V' in M such that V is contractible (respectively, nullhomologous) in V'.*

For we need only let K be a spine of V.

It is tempting to try to apply this result to a fake 3-sphere H. Let K_0 be a knot in H which is not contained in any 3-cell of H (see [1]). Then K_0 is contractible within a handlebody; we choose K_1 to be such a handlebody of minimal genus g_1. K_1 is contractible in H, so we may choose a handlebody K_2 of minimal genus g_2, and so on. The sequence $g_1, g_2, \cdots$ is nondecreasing. The Poincaré conjecture would follow from Haken's work [3] if we could show that the surfaces Bd K_i are incompressible in $H-K_0$. Unfortunately, while the map $\pi_1(\text{Bd } K_i)\to\pi_1(K_i-K_{i-1})$ is easily seen to be 1:1, very little can be said of the map $\pi_1(\text{Bd } K_i)\to\pi_1(H-K_i)$; after all, this same construction can be made in a Poincaré space (homology 3-sphere) if we replace "contractible" by "nullhomologous."

Let $L=k_1\cup k_2$ be an n-split link in an orientable 3-manifold M (see [7]), that is there exist 3-manifolds $V_0, V_1, \cdots, V_n$ such that $k_1\subset V_0\subset V_1\subset \cdots \subset V_n\subset M-k_2$ and the maps $H_1(k_1)\to H_1(V_0)$ and $H_1(V_i)\to H_1(V_{i+1})$ are trivial. Then k_1 is nullhomologous in a handlebody $K_0\subset V_0$. Since $H_1(K_0)\to H_1(V_1)$ is trivial, K_0 is nullhomologous within a handlebody K_1 in V_1. Continuing, we see that k_1 can be n-split from k_2 by a sequence of handlebodies. I believe this result can be used to show that n-splitting is an isotopy invariant of links.

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