CONGRUENCE RELATIONS IN DIRECT PRODUCTS

GRANT A. FRASER AND ALFRED HORN

Abstract. This paper studies conditions under which every congruence relation \( \theta \) in a direct product \( A \times B \) of algebras is of the form \( \theta_1 \times \theta_2 \), where \( \theta_1 \) and \( \theta_2 \) are congruence relations in \( A \) and \( B \) respectively. It is shown that for any equational class \( K \), every such \( \theta \) in every \( A \times B \) in \( K \) has this property if and only if \( K \) satisfies certain identities.

It is well known that if \( A \) and \( B \) are rings with unit element, then every ideal in \( A \times B \) is of the form \( I_1 \times I_2 \), where \( I_1 \) and \( I_2 \) are ideals in \( A \) and \( B \) respectively. In [1, Problem 40], G. Grätzer asks to characterize those equational classes \( K \) of algebras such that for any \( A, B \in K \), every congruence relation \( \theta \) in \( A \times B \) is of the form \( \theta_1 \times \theta_2 \), where \( \theta_1 \) and \( \theta_2 \) are congruence relations in \( A \) and \( B \) respectively. In Theorem 5 below, we shall show that \( K \) can be characterized by Malcev-type conditions, that is, by the existence of certain identities in \( K \).

We restrict our attention to algebras with finitary operations (possibly infinite in number). The set of congruence relations in \( A \) is denoted by \( C(A) \). If \( \theta \in C(A) \), then \( a_1 \theta a_2 \) denotes \( (a_1, a_2) \in \theta \). If \( \theta_1 \in C(A) \) and \( \theta_2 \in C(B) \), let \( \theta_1 \times \theta_2 \) be the relation \( \{(a_1, b_1), (a_2, b_2) : a_1 \theta_1 a_2 \text{ and } b_1 \theta_2 b_2\} \). Clearly \( \theta_1 \times \theta_2 \in C(A \times B) \). Let \( I_A = \{(a, a) : a \in A\} \) be the identity relation on \( A \), and \( U_A = A \times A \) be the universal relation on \( A \). If \( a_1, a_2 \in A \), let \( \theta(a_1, a_2) \) be the smallest congruence relation on \( A \) which contains \( (a_1, a_2) \).

In \( C(A \times B) \), we let \( \Pi_1 \) denote the kernel of the projection on \( A \). Clearly \( \Pi_1 = \{((a_1, b_1), (a_2, b_2)) : a_1 = a_2 \} = I_A \times U_B \); similarly let \( \Pi_2 = U_A \times I_B \). It is well known (see [1]) that \( C(A) \) is a lattice in which \( \theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2 \), and

\[
\theta_1 \vee \theta_2 = \bigcup_{n<\omega} (\rho_0 \circ \rho_1 \circ \cdots \circ \rho_n),
\]

where \( \rho_i = \theta_1 \) for even \( i \), and \( \rho_i = \theta_2 \) for odd \( i \).

**Lemma 1.** If \( \rho_1, \theta_1 \in C(A) \) and \( \rho_2, \theta_2 \in C(B) \), then

\[
(\rho_1 \times \rho_2) \vee (\theta_1 \times \theta_2) = (\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2).
\]

Received by the editors January 15, 1970.

AMS 1969 subject classifications. Primary 0830; Secondary 0245, 0250.

Key words and phrases. Congruence relation, direct product, equational class.

1 This research was supported in part by NSF grant GP-9044.

2 This problem has been solved independently by Tah-Kai Hu.

390
Proof. It is obvious that $\rho_1 \times \rho_2$ and $\theta_1 \times \theta_2$ are contained in $(\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2)$. Now let $(a, b) \in (\rho_1 \vee \theta_1) \times (\rho_2 \vee \theta_2)$. Then for some $m$, $a \in \phi_0 \circ \cdots \circ \phi_m$ and $b \in \psi_0 \circ \cdots \circ \psi_m$, where $\phi_i = \rho_1$, $\psi_i = \rho_2$ for even $i$, and $\phi_i = \theta_1$, $\psi_i = \theta_2$ for odd $i$ (since $\phi_0 \circ \cdots \circ \phi_n \subseteq \phi_0 \circ \cdots \circ \phi_{n+1}$); hence

$$(a, b) \in (\phi_0 \circ \cdots \circ \phi_m) \times (\psi_0 \circ \cdots \circ \psi_m) = (\phi_0 \times \psi_0) \circ \cdots \circ (\phi_m \times \psi_m) \subseteq (\rho_1 \times \rho_2) \vee (\theta_1 \times \theta_2).$$

Definition. If $\theta \in C(A \times B)$, we say $\theta$ has property $P$ if there exist $\theta_1 \in C(A)$, $\theta_2 \in C(B)$ such that $\theta = \theta_1 \times \theta_2$. We say $A \times B$ has property $P$ if every $\theta \in C(A \times B)$ has property $P$. If $K$ is a class of similar algebras, we say $K$ has property $P$ if for all $A, B \in K$, $A \times B$ has property $P$.

It is easily seen that $A \times B$ has property $P$ if and only if the map $(\theta_1, \theta_2) \rightarrow \theta_1 \times \theta_2$ is an isomorphism from $C(A) \times C(B)$ onto $C(A \times B)$.

Theorem 1. Let $\theta \in C(A \times B)$. Then the following are equivalent:

1. $\theta$ has property $P$.
2. $\Pi_2 \cap (\Pi_1 \vee \theta) \subseteq \theta$ and $\Pi_1 \cap (\Pi_2 \vee \theta) \subseteq \theta$.
3. For all $a, a_1, a_2 \in A$ and all $b, b_1, b_2 \in B$, $(a_1, b_1) \theta(a_2, b_2)$ implies $(a_1, b) \theta(a_2, b)$ and $(a, b_1) \theta(a, b_2)$.

Proof. (1) $\rightarrow$ (2): If $\theta = \theta_1 \times \theta_2$ then by Lemma 1,

$$\Pi_2 \cap (\Pi_1 \vee \theta) = \Pi_2 \cap ((I_A \times U_B) \vee (\theta_1 \times \theta_2)) = (U_A \times I_B) \cap (\theta_1 \times U_B) = \theta_1 \times I_B \subseteq \theta,$$

and similarly $\Pi_1 \cap (\Pi_2 \vee \theta) \subseteq \theta$.

(2) $\rightarrow$ (3): Suppose $(a_1, b_1) \theta(a_2, b_2)$. Since $(a_1, b) \Pi_1(a_1, b_1)$ and $(a_2, b_2) \Pi_1(a_2, b)$ we have $(a_1, b)(\Pi_1 \circ \theta \circ \Pi_1)(a_2, b)$. Since $\Pi_1 \circ \theta \circ \Pi_1 \subseteq \theta \circ \Pi_1$, it follows that

$$(a_1, b)(\Pi_2 \cap (\theta \cap \Pi_1))(a_2, b)$$

and by (2), $(a_1, b) \theta(a_2, b)$. Similarly $(a, b_1) \theta(a, b_2)$.

(3) $\rightarrow$ (1): Let

$$\theta_1 = \{(a_1, a_2) : \text{for some } b, (a_1, b) \theta(a_2, b)\},$$

$$\theta_2 = \{(b_1, b_2) : \text{for some } a, (a, b_1) \theta(a, b_2)\}.$$ 

Then by (3), $a \theta_1 a_2$ implies $(a_1, b) \theta(a_2, b)$ for all $b \in B$. It is very easy to verify that $\theta_1 \in C(A)$, $\theta_2 \in C(B)$ and $\theta = \theta_1 \times \theta_2$.

Corollary 1. If $C(A \times B)$ is a distributive lattice, then $A \times B$ has property $P$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Since $\Pi_1 \cap \Pi_3 = I_{A \times B}$, it is easily seen that (2) holds.

Corollary 1 was pointed out to us by Alfred Hales. He gave a different proof of the sufficiency part of the next theorem, from which Corollary 1 also follows.

**Theorem 2.** If $\theta \in C(A \times B)$, then $\theta$ has property $P$ if and only if (4)

$$\pi_1 \vee \theta \cap (\pi_2 \vee \theta) = \theta.$$ 

Proof. If $\theta = \theta_1 \times \theta_2$, then by Lemma 1,

$$\pi_1 \vee \theta \cap (\pi_2 \vee \theta) = [\pi_1 \times \theta_1] \cap [\pi_2 \times \theta_2] = \theta_1 \times \theta_2 = \theta.$$

Conversely, if (4) holds, then (2) is an immediate consequence.

**Theorem 3.** Let $A$ and $B$ be similar algebras. Then the following are equivalent:

1. $A \times B$ has property $P$.
2. For all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$, $\theta(a_1, a_2) \times \theta(b_1, b_2) = \theta((a_1, b_1), (a_2, b_2))$.
3. For all $a_1, a_2, a \in A$ and all $b_1, b_2, b \in B$, $((a_1, b), (a_2, b))$ and $((b_1, b_1), (b_2, b))$ are in $\theta((a_1, b), (a_2, b))$.

Proof. (1) $\Rightarrow$ (2): Clearly, $((a_1, b_1), (a_2, b_2)) \in \theta(a_1, a_2) \times \theta(b_1, b_2)$. Suppose $((a_1, b_1), (a_2, b_2)) \in \theta \in C(A \times B)$. By (1), $\theta = \theta_1 \times \theta_2$ for some $\theta_1 \in C(A)$, $\theta_2 \in C(B)$. Therefore $a_1 \theta a_2$ and $b_1 \theta b_2$. Hence $\theta = \theta_1 \times \theta_2 \supseteq \theta(a_1, a_2) \times \theta(b_1, b_2)$. This proves (2).

(2) $\Rightarrow$ (3): We have $((a_1, b), (a_2, b)) \in \theta(a_1, a_2) \times \theta(b_1, b_2) = \theta((a_1, b_1), (a_2, b_2))$ and similarly $((b_1, b_1), (b_2, b_2)) \in \theta((a_1, b_1), (a_2, b_2))$.

(3) $\Rightarrow$ (1): We show that (3) holds for every $\theta \in C(A \times B)$. Suppose $(a_1, b_1) \theta (a_2, b_2)$. Then $\theta \supseteq \theta((a_1, b), (a_2, b))$. Therefore by (3), (1) follows.

**Theorem 4.** Let $K$ be any class of similar algebras. Then $K$ has property $P$ if and only if (8)

$$\text{for all } A, B \in K, \text{ all } a_1, a_2 \in A, \text{ and all } b_1, b_2, b \in B, \theta((a_1, b), (a_2, b)) \in \theta((a_1, b_1), (a_2, b_2)).$$

Proof. The necessity of (8) is obvious by (7). To prove the sufficiency, we show that (7) holds for all $A, B \in K$. Let $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$. Then by (8)
Applying (8) to $B \times A$, we have

$$((a_1, b), (a_2, b)) \in \theta((a_1, b_1), (a_2, b_2)).$$

Using the canonical isomorphism of $B \times A$ with $A \times B$, we have

$$((a_1, b_2), (a, b_2)) \in \theta((a_1, b_1), (a_2, b_2)).$$

**Corollary 2.** If $K$ is an equational class such that for every $A \in K$ with two generators and for every $B \in K$ with three generators, $A \times B$ has property $P$, then $K$ has property $P$.

**Lemma 2.** If $u, v, c_0, c_1 \in A$, then $(u, v) \in \theta(c_0, c_1)$ if and only if for some $m \geq 1$, $n \geq 1$, there exist $(m+1)$-ary polynomials $p_1, \ldots, p_n$, elements $z_{ij}$ of $A$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and integers $k(1), \ldots, k(n)$ such that

$$k(i) = 0 \text{ or } 1 \quad \text{for } 1 \leq i \leq n,$$

$$u = p_1(c_{k(1)}, z_{i1}, \ldots, z_{im}), \quad v = p_n(c_{1-k(n)}, z_{n1}, \ldots, z_{nm}),$$

$$p_i(c_{1-k(i)}, z_{i1}, \ldots, z_{im}) = p_{i+1}(c_{k(i+1)}, z_{i+1,1}, \ldots, z_{i+1,m})$$

for $1 \leq i \leq n-1$.

**Proof.** This is a paraphrase of Theorem 3, p. 54 of [1]. As pointed out by G. Grätzer, we may take $m = n$ and $k(i) = 0$ for even $i$ and $k(i) = 1$ for odd $i$.

**Theorem 5.** Let $K$ be an equational class of algebras. Then $K$ has property $P$ if and only if for some $n \geq 1$ and some $m \geq 1$, there exist $(m+1)$-ary polynomials $p_1, \ldots, p_n$, binary polynomials $q_{ij}(x_0, x_1)$ and ternary polynomials $r_{ij}(x_0, x_1, x_2)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and integers $k(1), \ldots, k(n)$ which are 0 or 1 such that the following identities hold in all members of $K$:

$$x_0 = p_1(c_{k(1)}, z_{11}, \ldots, z_{1m}), \quad x_1 = p_n(c_{1-k(n)}, z_{n1}, \ldots, z_{nm}),$$

$$p_i(c_{1-k(i)}, z_{i1}, \ldots, z_{im}) = p_{i+1}(c_{k(i+1)}, z_{i+1,1}, \ldots, z_{i+1,m}),$$

$$1 \leq i \leq n - 1,$$

$$x_2 = p_1(c_{k(1)}, r_{11}, \ldots, r_{1m}) = p_n(c_{1-k(n)}, r_{n1}, \ldots, r_{nm}),$$

$$p_i(c_{1-k(i)}, r_{i1}, \ldots, r_{im}) = p_{i+1}(c_{k(i+1)}, r_{i+1,1}, \ldots, r_{i+1,m}),$$

$$1 \leq i \leq n - 1.$$

**Proof.** Suppose $K$ has property $P$. Let $A$ be the free $K$ algebra with two free generators $x_0, x_1$ and $B$ be the free $K$ algebra with three free generators $x_0, x_1, x_2$. Then by Theorem 4, $((x_0, x_2), (x_1, x_2))$
To obtain (9), we use Lemma 2, the fact that every element of $A \times B$ is of the form $(q(x_0, x_1), r(x_0, x_1, x_2))$, and that $\rho((u_0, v_0), \ldots, (u_m, v_m)) = (\rho(u_0, \ldots, u_m), \rho(v_0, \ldots, v_m))$ for every $(m+1)$-ary polynomial. Conversely, if we assume (9), then by substituting $x_0 = a_0, x_1 = a_1$ in the first three lines of (9) and $x_0 = b_0, x_1 = b_1$ and $x_2 = b$ in the last two lines of (9), we see by Lemma 2 that (8) holds.

We close with some examples and remarks. As pointed out before, the class $K_R$ of all rings with unit element has property $P$. More generally, if $K$ is such that there exist binary polynomials $+$ and $\cdot$, and constants (or polynomials which are constant in $K$) 0 and 1 such that the identities $x \cdot 1 = x + 0 = 0 + x = x$ and $x \cdot 0 = 0$ hold in $K$, then $K$ has property $P$. This follows from Theorem 5 with $n = 1, m = 2$, $p_1(x, y, z) = x \cdot y + z, q_{11} = 1, q_{12} = 0, r_{11} = 0, r_{12} = x_2$ and $k(1) = 0$.

Another example is the class $K_L$ of all lattices. In this case, the condition of Theorem 5 holds with $n = 1, m = 2$, $p_1(x, y, z) = (x \wedge y) \vee z, q_{11} = x_0 \wedge x_1, q_{12} = x_0 \wedge x_1, r_{11} = x_2, r_{12} = x_2$ and $k(1) = 0$. The fact that $K_L$ has property $P$ also follows from Corollary 1. Property $P$ also extends to any class $K$ (such as the class of lattice ordered groups) of algebras each of which is a lattice under some of its operations. There exist equational classes $K$ having property $P$ but such that $C(A \times B)$ is not distributive for some $A, B \in K$. As an example, let $K = K_R$ and $A = B =$ the ring of all polynomials in $x, y$ over the rationals. If $(p)$ denotes the principal ideal generated by $p$, then in $C(A), we have

$$(x + y) \cap ((x) \vee (y)) \subseteq ((x + y) \wedge (x)) \lor ((x + y) \wedge (y)).$$

We may define property $P$ in the obvious way for arbitrary direct products. However, the following theorem shows that property $P$ can only hold for essentially finite direct products.

**Theorem 6.** If $A = \prod_{i \in I} A_i$ has property $P$, then $A_i$ has one element for all but a finite number of $i$.

**Proof.** Let $\theta = \{(x, y): x(i) = y(i) \text{ for almost all } i\}$. Then $\theta \subseteq C(A)$. If $\theta = \bigcap_{i \in I} \theta_i$, then $\theta_i$ must be $U_{A_i}$ for all $i$, since $x \theta y$ whenever $x(j) = y(j)$ for all $j \in I, j \neq i$. Therefore $\theta = U_{A_i}$, from which it follows that $A_i$ is trivial for almost all $i$.

**References**


**University of California, Los Angeles, California 90024**